Invariant sets in the wind-tree model

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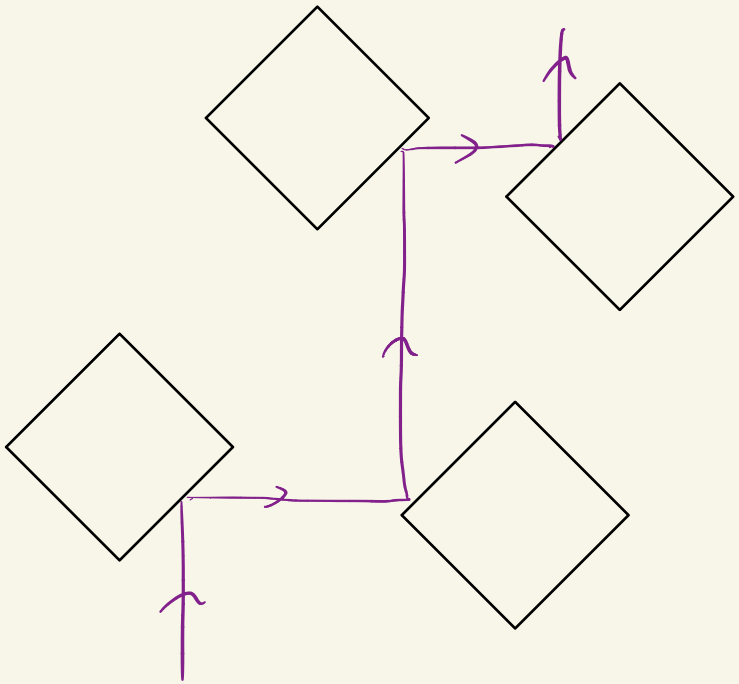
The wind-tree model

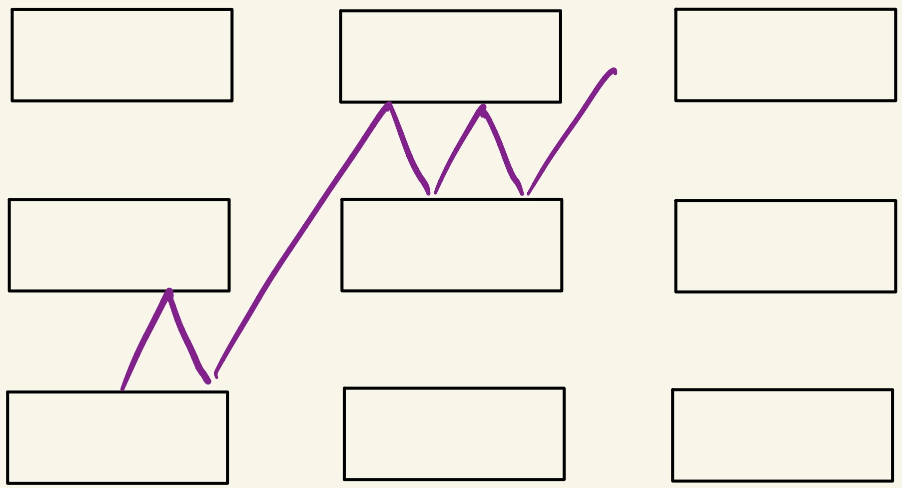
Paul & Tatiana Ehrenfest (1911)



- Randomly placed obstacles
- Billiard at 45° angles to obstacles

Hardy & Weber (1980)

- Parameters $a, b \in (0,1)$
- $a \times b$ rectangle obstacles placed at points of $\mathbb{Z}^2$
- Any direction



Previous Results

Recurrence Hubert - Lelièvre - Troubetzkoy, Avila - Hubert
G_δ - dense $\forall a, b$, a.e. θ

Diffusion Delecroix - Hubert - Lelièvre, Delecroix
$2/3$ for $\forall a, b$ a.e. θ exceptional θ

Delecroix - Zorich, Crovisier - Hubert - Lanneau - Pardo
more general obstacles   any diffusion rate occurs

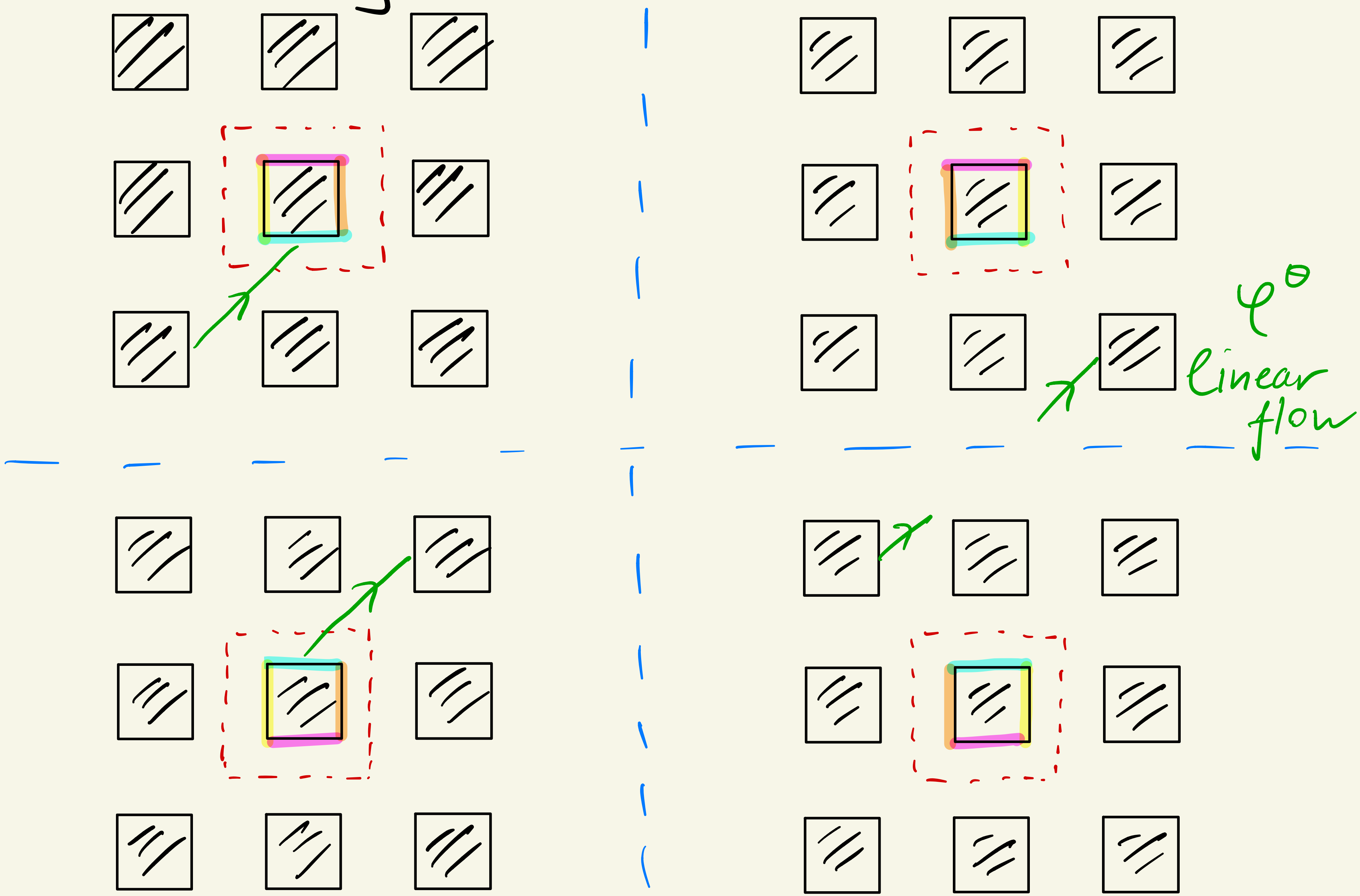
Non-ergodicity Frączek - Ulcigrai, Frączek - Hubert

Periodic directions Pardo

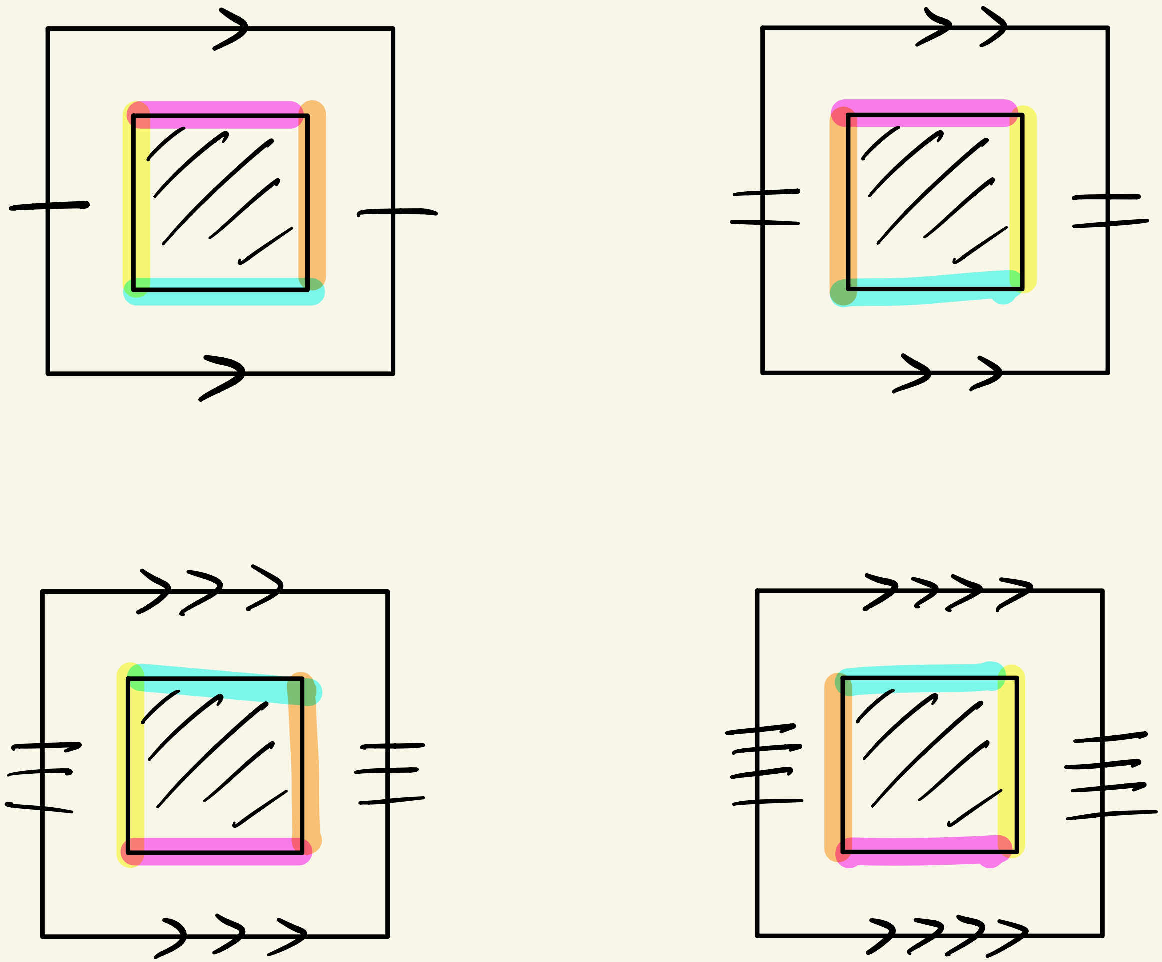
Aperiodic model Troubetzkoy, Málaga Sabogal - Troubetzkoy
recurrence   minimality, ergodicity

Unfolding

$\tilde{X}(a,b)$



$\tilde{X}(a,b)$ is a $\mathbb{Z}^2$-cover of a genus 5 translation surface $X(a,b)$



IETs and skew-products

- Poincaré section for φ^θ on $X(a,b)$ is an Interval Exchange Transformation $T: I \circlearrowleft$
  $I = [0,1)$
  - orientation-preserving piecewise isometry

- Section for φ^θ on $\tilde{X}(a,b)$ is a skew-product

$$T_{\phi_\gamma}: I \times \mathbb{Z}^2 \longrightarrow I \times \mathbb{Z}^2$$
$$(x, \underline{a}) \longmapsto (T(x), \underline{a} + \phi_\gamma(x))$$

where ϕ_γ is determined by the cover.

Non-ergodicity results

Thm (Frączek – Ulcigrai, 2014)

For every $(a,b) \in (0,1)^2$ and almost every θ, the linear flow φ^θ on $\widetilde{X}(a,b)$ is:

- not ergodic
- not transitive (no orbit is dense)

Thm (Frączek – Hubert, 2018)

Non-ergodicity / non-transitivity holds for more general wind-tree models / $\mathbb{Z}^d$-covers.

Open Questions

- We know typical directions are non-ergodic, so what are the ergodic components?

- We know typically no trajectories are dense, so what do their closures look like?

Self-similar case

- Suppose (a,b) such that there exists a pseudo-Anosov $\Psi : X(a,b) \circlearrowleft$
- Let θ be the contracted direction of Ψ

$\ulcorner r_{\frac{\pi}{2}-\theta} X(a,b)$ lies on a closed Teichmüller geodesic $\lrcorner$

Action of Ψ on $H_1(X(a,b),\mathbb{Z})$

- Let $c \in H_1(X(a,b),\mathbb{Z})$

  There are 3 cases:

  1.) (periodic) c is preserved by Ψ

  2.) (central) $(\Psi_*)^n c$ has sub-exponential growth

  3.) (unstable) $(\Psi_*)^n c$ has exponential growth

⚠ An integer vector can not be <u>stable</u>
(shrink exponentially)

Main result

The cover $\tilde{X}(a,b)$ is given by two classes $[\gamma_h], [\gamma_v] \in H_1(X(a,b), \mathbb{Z})$.

Thm A (T. '25+) If at least one of $[\gamma_h], [\gamma_v]$ is unstable, then all trajectories of φ^θ have Hdim < 2.

Other cases

<u>Thm</u> (T. '25) If both $[\gamma_h]$ and $[\gamma_v]$ are periodic, then φ^θ is ergodic on $\widetilde{X}(a,b)$.

(More: classification of ergodic measures – Maharam measures)

Still open: both central, central + periodic

Main Tool

Thm B (T. '25+)

X(a, b) self-similar, θ contracted direction

If $[\gamma_n]$ unstable, there exists $c \neq 0$, a continuous $\hat{h} : \tilde{X}(a,b) \to \mathbb{R}/c\mathbb{Z}$ invariant for the flow φ^θ.

Can replace with setting of Frączek - Hubert

Sketch of construction of invariant $\hat{h}$

1.) IET $T: I^{\circlearrowleft}$ Poincaré section for φ^θ on $X(a,b)$

2.) If $[\gamma_n]$ unstable, there is also a stable class α.

3.) [Marmi-Moussa-Yoccoz] Stable $\alpha \rightsquigarrow$ cont. coboundary ψ, i.e. for $\psi: I \to \mathbb{R}$, exists continuous $h: I \to \mathbb{R}$,

$$\psi(x) = h(T(x)) - h(x)$$

4.) Invariant function: $\hat{h}: I \times \mathbb{Z} \to \mathbb{R}/c\mathbb{Z}$ some c corresp to α

$$(x, a) \mapsto h(x) - a \pmod{c}$$

Idea of proof of Thm A

1.) Closures of trajectories of φ^0 on $\tilde{X}(a,b)$ are contained in level sets of $\hat{h}$.

2.) Suffices to consider level sets of h.

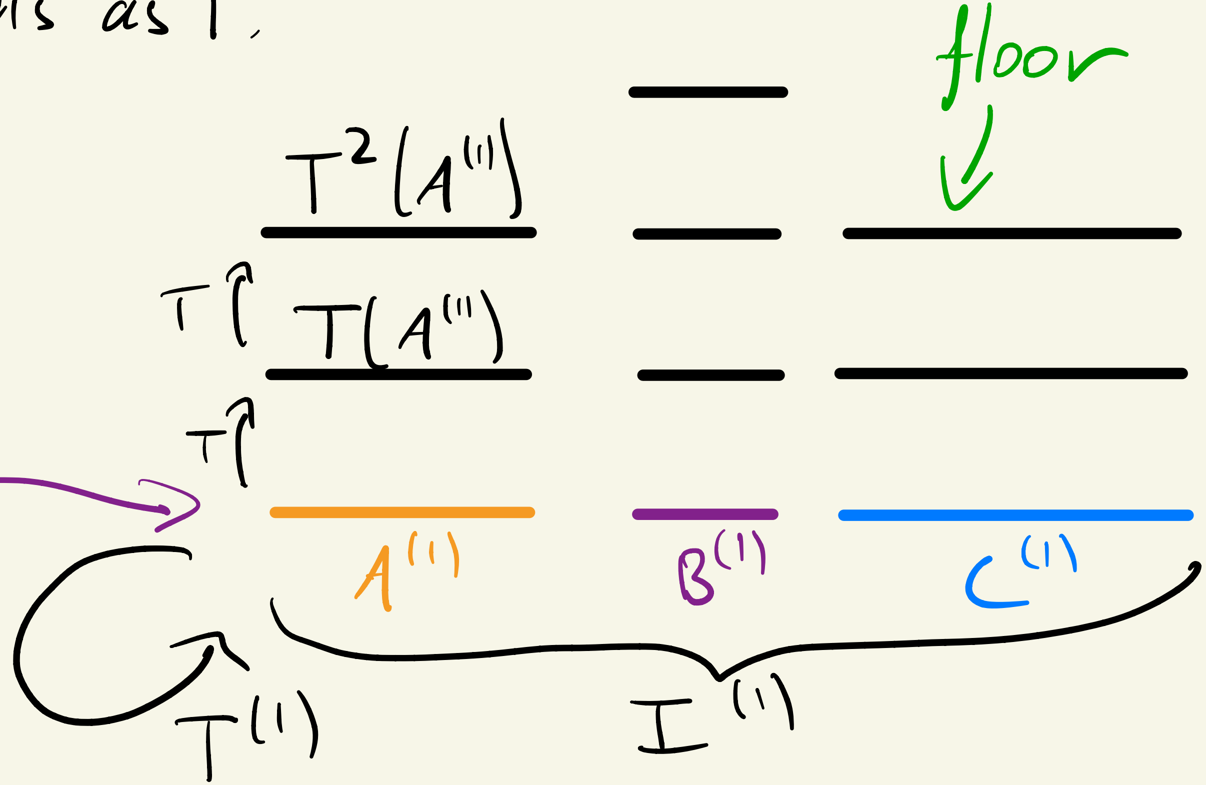
(Level set of $\hat{h}$ locally is finite union of level sets of h)

3.) Use adic coding for IET to find gaps in level set of h.

Towers for IETs

Rauzy-Veech induction produces $I \supseteq I^{(1)} \supseteq I^{(2)} \supseteq \dots$ such that for all k, $T^{(k)}$, the first return map of T to $I^{(k)}$, is also IET on same # intervals as T.

Rokhlin towers



Intervals of continuity of $T^{(1)}$

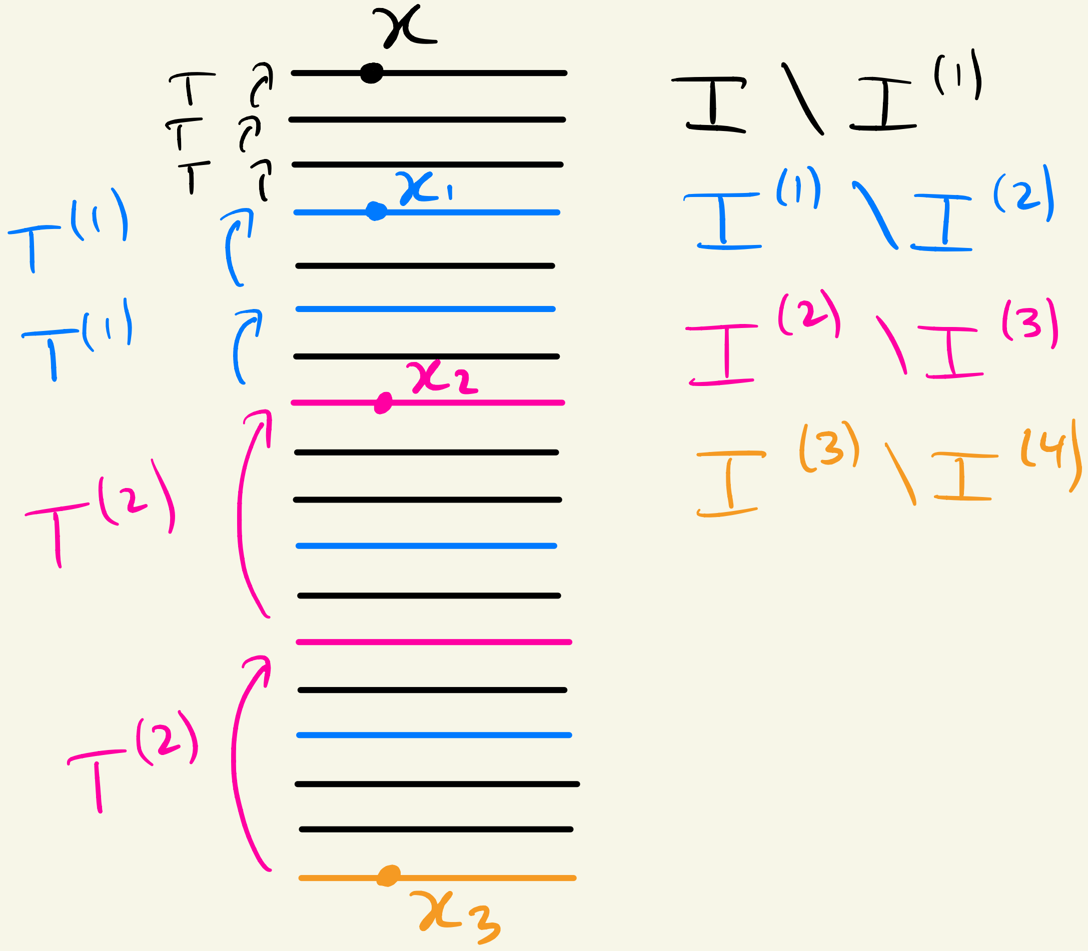
Problem

- T self-similar IET ($T^{(1)}$ is T up to rescaling)
- A corresponding Rauzy-Veech matrix
  ($A_{\alpha\beta} = \#\{\text{floors } F \text{ in tower over } I_\beta^{(1)} \text{ w/ } F \subset I_\alpha\}$)
- ψ piece-wise constant on I, $\psi \in \mathbb{R}^d$ vector
  $\psi_\alpha = \psi|_{I_\alpha}$. Assume $A^T \psi = \lambda \psi$, $0 < \lambda < 1$.
- h continuous, $h(T(x)) - h(x) = \psi(x)$

Q: What does level set $h^{-1}(z)$ look like?

Adic coding

- $n_k := \min\{n \geq 1 : T^{-n} \in I^{(k)}\}$
- $x_k := T^{-n_k}(x)$
- $x_k \in I^{(k)}_{\alpha_k}$
- $x_{k-1} = \left(T^{(k-1)}\right)^{\ell_k}(x_k)$



Code x by

$$\left((\alpha_1, \ell_1), (\alpha_2, \ell_2), \dots \right)$$

Gaps in level sets

Lem If $x \in [0,1]$ has coding $(e_1, e_2, e_3, \dots)$, then

$$h(x) = \sum_{k \geq 1} \lambda^{k-1} f(e_k).$$

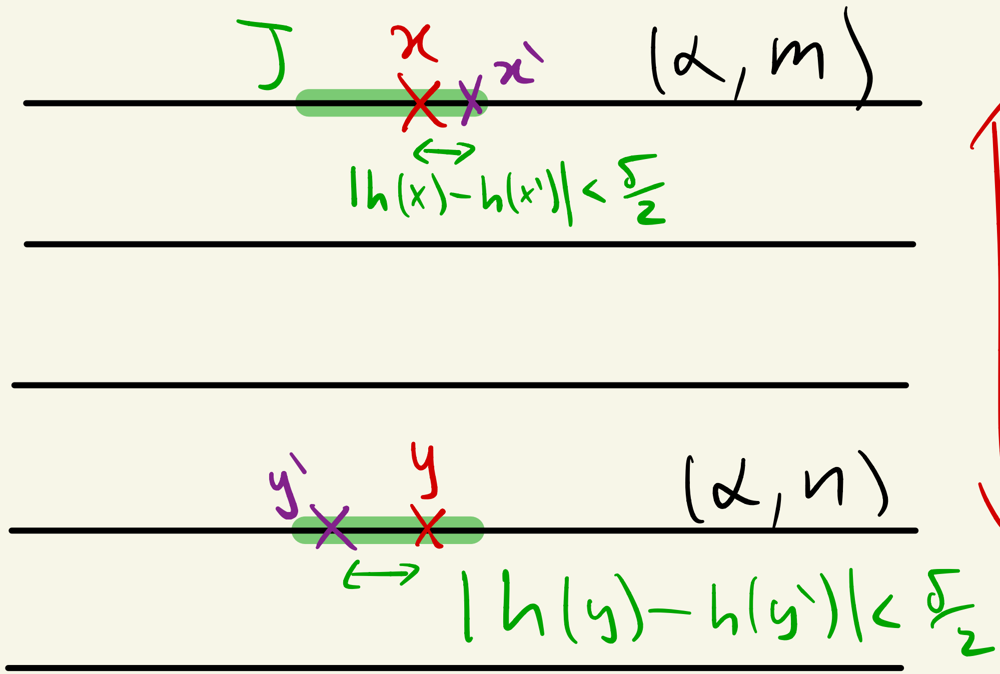
Cor If $f(e_1) \neq f(e_1')$, then for k large enough,

$$\forall x = (e_1, e_2, e_3, \dots, e_k, e_{k+1}, \dots)$$
$$x' = (e_1', e_2, e_3, \dots, e_k, e_{k+1}', e_{k+2}', \dots)$$

$h(x) \neq h(x')$.

$\Rightarrow \forall z$, one of the cylinder sets is a gap in $h^{-1}(z)$.

Gaps: picture



$$|h(x) - h(y)| =$$
$$= |f(\alpha, m) - f(\alpha, n)| = \delta > 0$$

k large enough

Need green interval J small enough for

$$\max_{x, x' \in J} |h(x) - h(x')| < \frac{\delta}{2}.$$

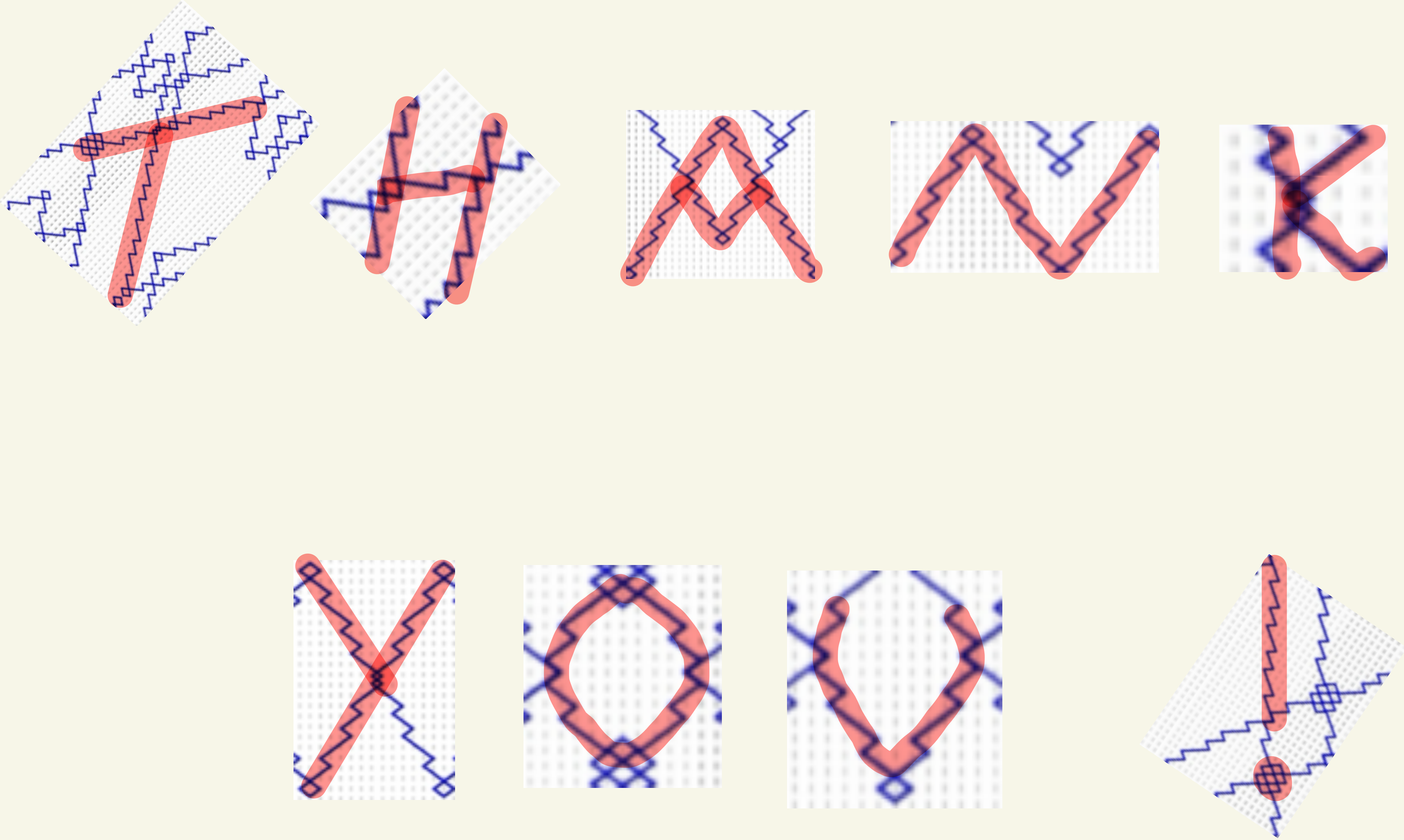
Cantor set

1.) Finitely many floors $\Rightarrow$ can choose uniform large enough k.

2.) Inductively construct Cantor set containing $h^{-1}(z)$ by finding gaps at all scales $\Rightarrow$ Hdim $h^{-1}(z) < 1$.

$$(e_1, e_2, \ldots, e_{nk}, e_{nk+1}, e_{nk+2}, \ldots, e_{(n+1)k}, e_{(n+1k)+1}, \ldots)$$

$$(e_1, e_2, \ldots, e_{nk}, \grave{e}_{nk+1}, e_{nk+2}, \ldots, e_{(n+1)k}, \grave{e}_{(n+1k)+1}, \ldots)$$


THANK
YOU!

Invariant function

- Suppose $H_1(S,\mathbb{Z}) = K \oplus K^\perp$ is invariant splitting for KZ cocycle
- Assume $\dim K = 2$, Lyapunov exponents on bundle over $K \otimes \mathbb{R}$ non-zero

Thm B If $[\gamma] \in K$, and $\tilde{S}_\gamma$ is the corresponding $\mathbb{Z}$-cover, then there is a continuous invariant function for φ^v on $\tilde{S}_\gamma$.

Define $\Phi : H_1(S, \mathbb{R}) \longrightarrow \mathbb{R}^n \simeq \left\{ \begin{array}{l} \text{piece-wise const.} \\ \text{functions on } I, \\ \text{const on } I_\alpha \end{array} \right\}$

$$c \longmapsto (\langle c, \xi_1 \rangle, \langle c, \xi_2 \rangle, \ldots)$$

Write $K = \operatorname{span}\{\gamma, \sigma\}$

Let $\alpha = \gamma + c\sigma \in K \otimes \mathbb{R}$ stable

$$\Rightarrow \quad \underbrace{\Phi(\alpha)}_{\substack{\psi \\ \text{coboundary}}} = \underbrace{\Phi(\gamma)}_{\substack{\phi_\gamma \\ \text{def. cover}}} + c\,\underbrace{\Phi(\sigma)}_{\in \mathbb{Z}^n}$$

Let h continuous transfer function,

$$h(T(x)) - h(x) = \psi(x)$$

Have: $\bullet\ \psi = \phi_\gamma + c\mathbb{Z}^n$

$\bullet\ h(T(x)) - h(x) = \psi(x)$

Define $\tilde{h} : I \times \mathbb{Z} \longrightarrow \mathbb{R}$

$(x, a) \longmapsto h(x) - a$

Then $\tilde{h}(T_{\phi_\gamma}(x,a)) - \tilde{h}(x,a)$

$$= h(T(x)) - (a + \phi_\gamma(x)) - h(x) + a$$

$$= \psi(x) - \phi_\gamma(x) \in c\mathbb{Z}$$

$\Rightarrow\ \hat{h} : I \times \mathbb{Z} \to \mathbb{R}/c\mathbb{Z}$ is T_{ϕ_γ}-invariant

$(x,a) \longmapsto \tilde{h}(x,a) \bmod c\mathbb{Z}$