Displaceability of Lagrangian Toric Fibres

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May 4, 2022

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1 Introduction

The Lagrangian displaceability problem asks whether a given Lagrangian L in a symplectic manifold (M, ω) can be *displaced* (moved off itself) by a Hamiltonian flow. This question has attracted a lot of attention and some powerful tools have been developed to answer it, most notably using Floer-theoretic approaches, such as for example those developed by Entov-Polterovich (e.g. [9]) and Fukaya-Oh-Ohta-Ono (e.g. [10, 11]). In general it is very hard to displace a Lagrangian, or to prove that it is non-displaceable, but in some special cases this can be done by elementary methods, and the presentation of two of these is the aim of this essay.

We focus, as much of the work on this problem has, on *symplectic toric manifolds*, which are symplectic manifolds admitting a *Hamiltonian action* by a half-dimensional real torus. Symplectic toric manifolds are good examples to study since they admit a combinatorial description in terms of *moment polytopes*, which allows much to be said about the manifolds using purely combinatorial tools. In particular, moment polytopes fully classify compact symplectic toric manifolds, whereas no such classification exists for general compact symplectic manifolds. We must also mention that toric geometry originated as an area of algebraic geometry and that toric varieties have been much studied from the algebraic perspective (see for example [7], which is a comprehensive reference on this theory.)

We will also pay particular attention to *monotone symplectic toric manifolds*, which are a special class of toric manifolds with a specific relation between their symplectic structure and topology. Monotone toric manifolds have a very specific form of moment polytope, and this will allow us to prove much stronger results about them than what can be said about symplectic toric manifolds in general.

For a toric manifold the generic orbits of the torus action – the *toric fibres* – are Lagrangian, and hence are a very natural choice of Lagrangians to study. These will be the Lagrangians that we will investigate the displaceability of.

The first method that we will discuss is McDuff's *method of probes* ([14]), which can be used to show that certain toric fibres are displaceable. A probe is a line segment that enters a moment polytope at an interior point of a facet in a direction integrally transverse to that facet. The method of probes is based on the following result:

Theorem 3.1.2. *Suppose M is a compact smooth symplectic toric manifold, with moment polytope $\Delta = \mu(M)$. Suppose F is a facet of Δ, $p = p_{F,\lambda}(w)$ a probe from a point $w \in \mathring{F}$. Then for points x lying on the probe p, less than halfway along p from w, the fibre $\mu^{-1}(x)$ is displaceable.*

For monotone toric manifolds, there is a combinatorial condition introduced by McDuff, called the *star Ewald condition*, which is equivalent to all except the central fibre being displaceable by probe.

Theorem 3.2.2. *If a monotone polytope Δ is star Ewald, then every point in $\mathring{\Delta} \setminus \{0\}$ is displaceable by probe.*

McDuff proved that all monotone polytopes of dimension ≤ 3 satisfy the star Ewald condition, and hence the following result:

Theorem 3.2.12. *For monotone smooth polytopes Δ of dimension ≤ 3, all points in $\mathring{\Delta}$ except the central point are displaceable by probe.*

We observe however that in the non-monotone case the method of probes is not as strong, and in section 3.3 we present an example from [14] of a 2-dimensional polytope with an open set of fibres that are non-displaceable by probe.

The second method that we will present is the symplectic reduction method of Abreu and Macarini ([3]), which is based on the following theorem:

Theorem 4.1.2. *Suppose that* (M_{red}, ω_{red}) *is a a symplectic reduction of* (M, ω)*, suppose* Δ *is a moment polytope of* M *and* $\Delta_{red} \subset \Delta$ *is the moment polytope of* M_{red}*. Suppose* $x \in \mathring{\Delta}_{red}$*. Let* T_x *the corresponding toric fibre of* M_{red} *and* $\tilde{T}_x$ *the corresponding toric fibre of* M*.*

If ψ^t *is a Hamiltonian flow on* M_{red}*, and* $\psi(T_x) \cap T_x = \emptyset$*, then there is a Hamiltonian flow* $\tilde{\psi}^t$ *on* M *with* $\tilde{\psi}(\tilde{T}_x) \cap \tilde{T}_x = \emptyset$*.*
Thus if T_x *is displaceable, then* $\tilde{T}_x$ *is also displaceable.*

Thus this method allows us to deduce the non-displaceability of fibres in the symplectic reduction of a manifold M if there are some non-displaceable fibres in M. Since we can construct complicated toric manifolds from much simpler ones by symplectic reduction, this method allows us to construct more complicated examples of non-displaceable fibres from simpler ones. Using a result of Cho and Poddar ([6]) which states the non-displaceability of a fibre of weighted projective spaces, and expressing every monotone toric manifold as a reduction of a weighted projective space, Abreu and Macarini prove:

Theorem 4.2.4. *For any monotone symplectic toric manifold* M*, the central fibre is non-displaceable.*

Together with Theorem 3.2.12 we thus obtain a complete answer to the question of displaceability of toric fibres for monotone toric manifolds of dimension ≤ 6, namely that in this case there is a unique non-displaceable fibre, the central fibre.

We finish the presentation of the method of Abreu and Macarini with some examples of applications of the method in cases where a toric manifold can be expressed as a reduction of a product of simpler toric manifolds. We present a few two-dimensional examples from [3] illustrating some interesting phenomena, such as the *collide-and-scatter phenomenon* and an example with a segment of non-displaceable fibres, and add two new examples illustrating these phenomena in three dimensions.

2 Background

Here we discuss the background theory of symplectic toric manifolds, and give some results and constructions that are fundamental for the theory. In particular, we first define symplectic toric manifolds and moment polytopes, the associated combinatorial objects that we will mostly work with. We then talk about symplectic reduction, a method of constructing (often more interesting) toric manifolds out of existing one. We then give (one direction of) a proof of Delzant's theorem, which establishes an equivalence between toric manifolds and moment polytopes. After that we prove some auxiliary results about moment polytopes that will be necessary in following sections, and also discuss a special class of symplectic toric manifolds called monotone, which will be the setting of many of our results. This section is mostly based on [8], which contains the relevant references.

We start by defining what it means for a Lagrangian to be displaceable by Hamiltonian isotopy, since this is the basic property that we are investigating.

Definition 2.0.1. Let (M, ω) a symplectic manifold. A (possibly time-dependent) **symplectic vector field** on M is a vector field X such that the 1-form $\iota_X \omega$ is closed. If further $\iota_X \omega$ is exact, so $\iota_X \omega = \mathrm{d}H$ for some $H : M \to \mathbb{R}$ (or if X is time-dependent then $H : M \times [0, 1] \to \mathbb{R}$), then X is a **Hamiltonian vector field** and H its **Hamiltonian function**.

A Hamiltonian vector field X induces a **Hamiltonian flow** φ^t. We say that a Lagrangian $L \subset M$ is **displaceable by a Hamiltonian flow** if there is such a Hamiltonian flow φ^t such that $L \cap \varphi^1(L) = \emptyset$.

2.1 Symplectic toric manifolds

Definition 2.1.1. Let (M, ω) a symplectic manifold, G a Lie group with Lie algebra $\mathfrak{g}$. A **symplectic action** of G on M is a homomorphism $\psi : G \to \mathrm{Symp}(M, \omega)$ from G into the group of symplectomorphisms of M.

Denote $\psi_g = \psi(g) \in \mathrm{Symp}(M, \omega)$.

The action is **Hamiltonian** if there is a **moment map** $\mu : M \to \mathfrak{g}^*$ which satisfies:

(i) For $X \in \mathfrak{g}$, let $X^{\#}$ be the vector field on M generated by the action of the one-parameter subgroup $\{\exp(tX) : t \in \mathbb{R}\}$. Let $\mu^X : M \to \mathbb{R}$ be the component of μ along X, $\mu^X(p) = \langle \mu(p), X \rangle$. Then for any $X \in \mathfrak{g}$, μ^X is the Hamiltonian for the vector field $X^{\#}$:

$$\mathrm{d}\mu^X = \iota_{X^{\#}}\omega.$$

(ii) The moment map is G-equivariant, with respect to the co-adjoint action on $\mathfrak{g}$:

$$\forall g \in G,\ \mu \circ \psi_g = \mathrm{Ad}_g^* \circ \mu.$$

The data (M, ω, G, μ) is called a **Hamiltonian G-space**.

Definition 2.1.2. A **symplectic toric manifold** is a connected symplectic manifold (M^{2n}, ω) with an effective Hamiltonian action of a real n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. (Where effective means that every non-identity element acts non-trivially.)

For a toric manifold, $\mathfrak{g} = \mathbb{R}^n$, which has a standard basis $X_1, \dots, X_n$. Then writing $\mu : M \to (\mathbb{R}^n)^*$ as $(\mu_1, \dots, \mu_n)$, condition (i) of Defenition 2.1.1 says that μ_i is a Hamiltonian function for the action induced by X_i. As $\mathbb{T}^n$ is commutative, the co-adjoint action is trivial, so condition (ii) says that μ is constant on the torus orbits.

Example 2.1.3. Consider $M = \mathbb{C}^n$ with the normalised standard symplectic form $\omega = \frac{-i}{\pi}\sum_{i=1}^n \mathrm{d}z \wedge \mathrm{d}\bar{z}$. $\mathbb{T}^n$ acts on $\mathbb{C}^n$ via $(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) = (e^{2\pi i t_1} z_1, \dots, e^{2\pi i t_n} z_n)$.

Then for the standard basis $(X_1, \dots, X_n)$ of the Lie algebra $\mathbb{R}^n$ of the torus, $X_j^{\#} = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0}(z_1, \dots, e^{2\pi i t} z_j, \dots, z_n)$. Separating the tangent vector into its holomorphic and antiholomorphic parts, we get $\frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} e^{2\pi i t} z = i\pi\left(z\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \bar{z}}\right)$.

Hence $X_j^{\#} = i\pi(z_j\frac{\partial}{\partial z_j} - \bar{z}_j\frac{\partial}{\partial \bar{z}_j})$, and hence $\iota_{X_j^{\#}}\omega = (z_j \mathrm{d}\bar{z}_j + \bar{z}_j \mathrm{d}z_j) = \mathrm{d}|z_j|^2$. Hence the moment map is

$$\mu = (|z_1|^2, \dots, |z_n|^2).$$

We see that here the image of the moment map is the positive orthant in $\mathbb{R}^n$.
(Note that we can add any constant we like to μ as it doesn't affect its differential, and so we can translate the image $\mu(M)$ as we like. Scaling it by a real factor corresponds to scaling the symplectic form, so we could have obtained the negative orthant if we took $-\omega$ instead.) $\Diamond$

A foundational result of Atiyah ([4]) and Guillemin-Sternberg ([12]) says that for compact symplectic toric manifolds, the image of the moment map is a convex polytope. It is known as the **moment polytope**, and we will denote it by $\Delta := \mu(M)$.

Theorem 2.1.4. (Atiyah-Guillemin-Sternberg)

For a compact connected symplectic toric manifold M with moment map μ,

1. *the levels of μ are connected,*
2. *the image of μ is convex,*
3. *the image of μ is the convex hull of the images of the fixed points of the torus action.*

We shall see that if $x \in \mathring{\Delta}$, then $\mu^{-1}(x)$ is a Lagrangian orbit of the torus action, a **Lagrangian toric fibre**. We will investigate the displaceability of these fibres.

Example 2.1.5. Consider S^2 with cylindirical coordinates (θ, z) and the standard symplectic form $\omega = \mathrm{d}\theta \wedge \mathrm{d}z$. Consider the $\mathbb{T}^1 = S^1$ action $e^{i\phi} \cdot (\theta, z) = (\theta + \phi, z)$, which rotates the sphere about the vertical axis. If X is the unit vector spanning the Lie algebra $\mathbb{R}$ of $\mathbb{T}^1$, the induced vector field $X^\#$ is $\frac{\partial}{\partial\theta}$. Hence $\iota_{X^\#}\omega = \mathrm{d}z$.

Now if $\mu : S^2 \to \mathbb{R}^*$ is $\mu(\theta, z) = z$, then $\mathrm{d}\mu = \mathrm{d}z = \iota_{X^\#}\omega$. Since z is constant on the S^1-orbits, μ is a moment map for this action. The image of μ is the closed interval [-1,1], the convex hull of the images of the north and south poles, which are the fixed points of the action.

The preimage of a point $x \in \Delta$ is a latitude $L = \mu^{-1}(x) = \{z = x\}$, a Lagrangian toric fibre. For $x \neq 0$, L can be displaced by a rotation by π around the x-axis (if we think of S^2 as the unit sphere in $\mathbb{R}^3$), which is a Hamiltonian flow. However for $x = 0$, L is the equator. Hamiltonian flows are volume preserving, hence for any Hamiltonian symplectomorphism ϕ, $\phi(L)$ divides the sphere into two sections of equal area. Then neither section can lie entirely within the northern or southern hemisphere, so $\phi(L)$ can not lie entirely within one of the two hemispheres, and so $L \cap \phi(L) \neq \emptyset$.

Thus in this case there is a single non-displaceable fibre corresponding to $0 \in \Delta$, which we will see is an example of a more general phenomenon for a class of toric manifolds called monotone.

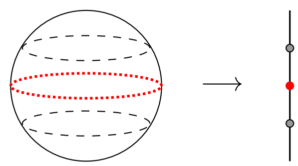


Figure 1: The moment map for the S^1 action on S^2 is the projection onto the z-axis. The non-displaceable fibre (the equator) is shown in red.

$\diamondsuit$

2.2 Symplectic reduction

Symplectic reduction is a way of, given a Hamiltonian G-space, obtaining new symplectic manifolds of lower dimension. It is particularly useful in the case of toric manifolds as the new manifold we obtain will again be toric. Starting with (M^{2N}, ω, G, μ), where G has dimension $N - n$, we will first take the submanifold $\mu^{-1}(c)$ for a regular value c, and then quotient it by the action of G to obtain $M_{\text{red}} = \mu^{-1}(c)/G$. Each of these steps reduces the dimension by $N - n$ so M_{red} has dimension $2n$.

The tangent space at $[p] \in M_{\text{red}}$ is $\mathrm{T}_p\mu^{-1}(c)/\mathrm{T}_p\mathcal{O}_p$, where $\mathcal{O}_p$ is the G-orbit of p. We will see that $\mathrm{T}_p\mu^{-1}(c) = (\mathrm{T}_p\mathcal{O}_p)^\omega$, and so there is a natural symplectic form ω_{red} on M_{red} induced from ω:

Lemma 2.2.1. *If (V, ω) is a symplectic vector space, I an isotropic subspace, then ω induces a symplectic form Ω on I^ω/I.*

Proof. For $u, v \in I^\omega$, define $\Omega([u],[v]) = \omega(u, v)$. Then

- Ω is well-defined: for any other representatives $u + i, v + j$ of $[u], [v]$ with $i, j \in I$, we have

$$\omega(u + i, v + j) = \omega(u, v) + \omega(u, j) + \omega(i, v) + \omega(i, j) = \omega(u, v).$$

  The second and third terms vanish as $u, v \in I^\omega$ and the last term as I is isotropic.

- Ω is automatically closed: if $\pi : I^\omega \to I^\omega/I$ is the projection, then $\Omega = \pi_*\omega$, and $\mathrm{d}(\pi_*\omega) = \pi_*(\mathrm{d}\omega) = 0$.

- Ω is non-degenerate: suppose $\Omega([u],[v]) = 0$ for all $[v] \in I^\omega/I$. Then $u \in I^\omega$ satisfies $\omega(u, v) = 0$ for all $v \in I^\omega$, hence $u \in (I^\omega)^\omega = I$, and so $[u] = 0$.

$\square$

The formal statement of the existence of symplectic reduction is:

Theorem 2.2.2. (Marsden-Weinstein-Meyer) *Let (M, ω, G, μ) a Hamiltonian G-space with G a compact Lie group. Suppose that c is a regular value of μ and that G acts freely on $Z = \mu^{-1}(c)$. Then:*

- *$M_{red} = Z/G$ is a manifold,*
- *$\pi : Z \to M_{red}$ is a principal G-bundle,*
- *there is a symplectic form ω_{red} on M_{red} such that $i^*\omega = \pi^*\omega_{red}$ where $i : Z \hookrightarrow M$ is inclusion.*

We will prove the following statement, which is the case for symplectic toric manifolds. A full proof of the Marsden-Weinstein-Meyer theorem is given for example in [8].

Theorem 2.2.3. (Symplectic reduction for toric manifolds) *Let (M^{2N}, ω) a symplectic toric manifold with a Hamiltonian $\mathbb{T}^N$ action generated by moment map $\mu : M \to (\mathbb{R}^N)^*$. Suppose K is an $(N-n)$-dimensional subtorus, with $k : \mathbb{R}^{N-n} \hookrightarrow \mathbb{R}^N$ the inclusion of Lie algebras. Let $\mu_K = k^* \circ \mu$, a moment map for the K-action on M. Suppose 0 is a regular value of μ_K, and assume that K acts freely on $Z := \mu_K^{-1}(0)$. Then we can define:*

- *a $2n$-manifold $M_{red} := Z/K$,*
- *a symplectic form ω_{red} on M_{red} such that $\pi^*\omega_{red} = i^*\omega$ where $\pi : Z \to M_{red}$ is the projection, $i : Z \hookrightarrow M$ the inclusion,*
- *a moment map μ_{red} for the action of $\mathbb{T}^n = \mathbb{T}^N/K$ on M_{red} satisfying the commutative diagram:*

$$\begin{array}{ccccc} Z & \overset{i}{\hookrightarrow} & M & \overset{\mu}{\longrightarrow} & \Delta \subset (\mathbb{R}^N)^* \\ \Big\downarrow{\scriptstyle \pi} & & & & \Big\uparrow{\scriptstyle j} \\ M_{\text{red}} & & \xrightarrow{\quad \mu_{\text{red}} \quad} & & \Delta_{\text{red}} \subset (\mathbb{R}^n)^* \end{array}$$

where j is the map $j : (\mathbb{R}^n)^* \xrightarrow{\cong} \ker k^* \subset (\mathbb{R}^N)^*$.

Thus we obtain a new toric manifold $(M_{red}, \omega_{red}, \mathbb{T}^n, \mu_{red})$.

Proof. Without loss of generality take K to be the first $N-n$ coordinates of $\mathbb{T}^N$ (we can always change coordinates to make this happen). Then $k : \mathbb{R}^{N-n} \to \mathbb{R}^N$ is just the inclusion of the first $N-n$ coordinates. Observe that $\mu_K = k^* \circ \mu$ is the moment map for the action of K. Suppose that 0 is a regular value of μ_K and let $Z = \mu_K^{-1}(0)$, a $(N+n)$-submanifold of M. Assuming that K acts freely on Z, $M_{\text{red}} := Z/K$ is a smooth $2n$-manifold.

Let $p \in Z$, denote by $[p]$ the image $\pi(p) \in M_{\text{red}}$. Then $\mathrm{T}_{[p]}M_{\text{red}} = \mathrm{T}_pZ/\mathrm{T}_p\mathcal{O}_p$ where $\mathcal{O}_p$ is the K-orbit through p. Now $\mathrm{T}_p\mathcal{O}_p = \{X_p^{\#} : X \in \mathbb{R}^{N-n}\}$, and $\mathrm{T}_pZ = \ker(\mathrm{d}\mu_K)_p$.

Claim.

$$(T_p\mathcal{O}_p)^{\omega_p} = \ker(d\mu_K)_p.$$

Proof. For $v \in \mathrm{T}_pM$, since μ_K is a moment map for the K action, we have $\langle (\mathrm{d}\mu_K)_p(v), X \rangle = \omega_p(X_p^{\#}, v)$ for $X \in \mathbb{R}^{N-n}$. Hence

$$\begin{aligned} v \in \ker(\mathrm{d}\mu_K)_p &\iff \forall X \in \mathbb{R}^{N-n} \quad \langle (\mathrm{d}\mu_K)_p(v), X\rangle = 0 \\ &\iff \forall X \in \mathbb{R}^{N-n} \quad \omega_p(X_p^\#, v) = 0 \\ &\iff \forall w \in \mathrm{T}_p\mathcal{O}_p \quad \omega_p(v, w) = 0 \\ &\iff v \in (\mathrm{T}_p\mathcal{O}_p)^{\omega_p}. \end{aligned}$$

□

Now as μ_K is constant on K-orbits, $\mathrm{T}_pZ \supset \mathrm{T}_p\mathcal{O}_p$, and hence

$$(\mathrm{T}_p\mathcal{O}_p)^{\omega_p} = \ker(\mathrm{d}\mu_K)_p = \mathrm{T}_pZ \supset \mathrm{T}_p\mathcal{O}_p.$$

Thus $\mathrm{T}_p\mathcal{O}_p$ is isotropic and so Lemma 2.2.1 gives us a symplectic form ω_{red} on M_{red}, defined by $\omega_{\mathrm{red}}([u],[v]) = \omega(u,v)$, where $[u] = \mathrm{d}\pi(u)$.

<u>The torus action on the reduced manifold</u>
Let $\mathbb{T}^n = \mathbb{T}^N/K$ be the last n coordinates of $\mathbb{T}^N$. Then $\mathbb{T}^n$ acts on M_{red} via $[\theta]\cdot[p] = [\theta\cdot p]$. Indeed, this is well-defined: suppose $[\theta'] = [\theta]$, so $\theta' = k_1 + \theta$, and $[p'] = [p]$, so $p' = k_2 \cdot p$ for some $k_1, k_2 \in K$.
Then $[\theta'\cdot p'] = [(\theta + k_1)\cdot(k_2\cdot p)] = [(k_1+k_2)\cdot(\theta\cdot p)] = [\theta\cdot p]$.

Let $j : (\mathbb{R}^n)^* \to (\mathbb{R}^N)^*$ be the map $(x) \to (0,x)$, and let $\mathrm{pr} : (\mathbb{R}^N)^* \to (\mathbb{R}^n)^*$ be the projection onto the last n coordinates, so $\mathrm{pr}\circ j$ is the identity on $(\mathbb{R}^n)^*$.
Define $\mu_{\mathrm{red}}([p]) = \mathrm{pr}\circ\mu(p)$. Thus if for $p \in Z$ we have $\mu(p) = (0,y) \in (\mathbb{R}^N)^* = (\mathbb{R}^{N-n})^* \times (\mathbb{R}^n)^*$, then $\mu_{\mathrm{red}}([p]) = y \in (\mathbb{R}^n)^*$.

Let us check that this is a moment map for the $\mathbb{T}^n$ action.

(i) Let X_i the standard basis for $(\mathbb{R}^N)^*$. For $i > N-n$, let $X_i^\#$ the induced vector field on M_{red}, $\tilde{X}_i^\#$ the induced vector field on M. Since $[\exp(tX)]\cdot[p] = [\exp(tX)\cdot p]$, we have

$$X_i^\# = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}([\exp(tX_i)]\cdot[p]) = \left[\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(\exp(tX)\cdot p)\right] = \left[\tilde{X}_i^\#\right].$$

Thus $\forall [v] \in \mathrm{T}_{[p]}M_{\mathrm{red}}$, we have:

$$\omega_{\mathrm{red}}\left(X_i^\#, [v]\right) = \omega\left(\tilde{X}_i^\#, v\right) = \mathrm{d}\mu_i(v) = \mathrm{d}(j\circ\mu_{\mathrm{red}}\circ\pi)_i(v) = \mathrm{d}(\mu_{\mathrm{red}})_i(\mathrm{d}\pi(v)) = \mathrm{d}(\mu_{\mathrm{red}})_i([v]).$$

Thus for all $X \in \mathbb{R}^n$, we have $\iota_{X^\#}\omega_{\mathrm{red}} = \mathrm{d}\mu_{\mathrm{red}}^X$.

(ii) μ_{red} is $\mathbb{T}^n$-invariant since μ is:

$$\mu_{\mathrm{red}}([\theta]\cdot[p]) = \mu_{\mathrm{red}}([\theta\cdot p]) = \mathrm{pr}\circ\mu(\theta\cdot p) = \mathrm{pr}\circ\mu(p) = \mu_{\mathrm{red}}([p]).$$

Thus indeed $(M_{\mathrm{red}}, \omega_{\mathrm{red}}, \mathbb{T}^n, \mu_{\mathrm{red}})$ is a symplectic toric manifold. □

Remark 2.2.4. Note that by identifying $(\mathbb{R}^n)^*$ with its image under j, we can see the reduced moment polytope Δ_{red} as a subset of the original moment polytope Δ. Further, suppose $x \in \Delta \cap \mathrm{im}j = \Delta \cap \ker k^*$. Then $\mu^{-1}(x) \subset \mu_K^{-1}(0) = Z$, and so $x \in \mathrm{im}(j\circ\mu_{red})$. Thus in fact $\Delta_{\mathrm{red}} = \Delta \cap \ker k^*$, so the reduced moment polytope is a section of the original moment polytope by a plane. ◊

Example 2.2.5. Start with $\mathbb{C}^{n+1}$ with the standard $\mathbb{T}^{n+1}$ action as in Example 2.1.3. Let $K \cong \mathbb{T}$ be generated by $(1,1,1,\ldots)$, so the action is diagonal: $t\cdot z = \left(e^{2\pi i t}z_0, \ldots, e^{2\pi i t}z_n\right)$. Change coordinates on $\mathbb{T}^{n+1}$ so that K is the first coordinate factor – this corresponds to taking the basis $\{\sum_{i=0}^n X_i, X_1, \ldots, X_n\}$ for $\mathbb{R}^{n+1}$. Then the action of $\mathbb{T}^{n+1}$ is $(t,\theta_1,\ldots,\theta_n)\cdot(z_0,\ldots,z_n) = \left(e^{2\pi i t}z_0, e^{2\pi i(t+\theta_1)}z_1, \ldots, e^{2\pi i(t+\theta_n)}z_n\right)$, and $K = \mathbb{T}^1 \times \{0\}$.

Then a moment map is $\mu_K = \sum_{i=0}^n |z_i|^2 - 1$, and for $i \geq 1$, $\mu_i = |z_i|^2 - 1$.

Let $Z = \mu_K^{-1}(0) = S^{2n+1} \subset \mathbb{C}^{n+1}$. Then K acts freely on Z and $M_{\text{red}} = S^{2n+1}/S^1 = \mathbb{C}\mathrm{P}^n$, with ω_{red} the standard Fubini–Study symplectic form. The moment map is

$$\mu_{\text{red}}([z_0 : \ldots : z_n]) = \text{pr} \circ \mu \left(\frac{z_0}{\sqrt{\sum |z_i|^2}}, \ldots, \frac{z_n}{\sqrt{\sum |z_i|^2}} \right) = \left(\frac{|z_1|^2}{\sum |z_i|^2}, \ldots, \frac{|z_n|^2}{\sum |z_i|^2} \right).$$

Thus a moment polytope for $\mathbb{C}\mathrm{P}^n$ is the set $\{x \in \mathbb{R}^n : 0 \leq x_i \leq 1, \sum x_i \leq 1\}$. For example for $\mathbb{C}\mathrm{P}^2$ it is a right-angled triangle, with the vertices corresponding to the torus fixed points $[1:0:0], [0:1:0], [0:0:1]$.

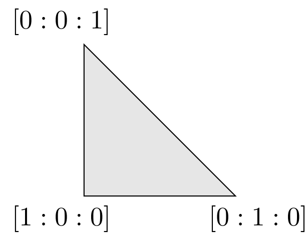


Figure 2: Moment polytope for $\mathbb{C}\mathrm{P}^2$.

◊

2.2.1 Reduction in stages

Suppose G is a product of two compact Lie groups, $G = G_1 \times G_2$. Then instead of reducing directly by G we can first reduce by G_1 and then by G_2. Luckily we get the same result both ways.

Specifically: suppose (M, ω, G, ψ) is Hamiltonian with $G = G_1 \times G_2$ a product of two compact Lie groups. Then the Lie algebras are $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, so we can write $\psi : M \to \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*$ as $\psi = (\psi_1, \psi_2)$ with $\psi_i : M \to \mathfrak{g}_i^*$. Suppose that G_1 acts freely on $Z_1 := \psi_1^{-1}(0)$. Then we obtain a reduced space (M_1, ω_1). Let $\iota_1 : Z_1 \hookrightarrow M$ be the inclusion map, $p_1 : Z_1 \to M_1$ the projection map.

Now as the G_2 action commutes with the G_1 action, G_2 acts on Z_1, and since G_2 preserves ω, it preserves ω_1, so the action is symplectic.

Further, as G_1 preserves ψ_2, it also preserves $\psi_2 \circ \iota_1 : Z_1 \xhookrightarrow{\iota_1} M \xrightarrow{\psi_2} \mathfrak{g}_2^*$. Then $\psi_2 \circ \iota_1$ is constant on G_1-orbits on Z_1, and hence $\psi_2 \circ \iota_1$ descends to a map $\mu_2 : M_1 \to \mathfrak{g}_2^*$, such that the following diagram commutes:

$$\begin{array}{ccc} Z_1 & \xhookrightarrow{\iota_1} & M \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \psi_2} \\ M_1 & \xrightarrow{\mu_2} & \mathfrak{g}_2^* \end{array}$$

Proposition 2.2.6. *In the setup above, μ_2 is a moment map for the action of G_2 on M_1.*

Further, suppose that G acts freely on $\psi^{-1}(0)$. Then G_2 acts freely on $\mu_2^{-1}(0)$ and the reduced spaces $\mu_2^{-1}(0)/G_2$ and $\psi^{-1}(0)/G$ are symplectomorphic.

Proof. Take a $X \in \mathfrak{g}_2$. Since the G_2 action preserves Z_1, the argument from the proof of Theorem 2.2.3 shows that the induced vector field $X^\#$ on M_1 is equal to $(p_1)_*(\tilde{X}^\#)$ where $\tilde{X}^\#$ is the induced vector field on M. So we see that for vectors $[v] \in \mathrm{T}M_1$

$$\iota_{X^\#}\omega_1([v]) = \iota_{\tilde{X}^\#}\omega(v) = \mathrm{d}\psi_2^X(v) = \mathrm{d}(\mu_2 \circ p_1)^X(v) = \mathrm{d}\mu_2^X([v]).$$

As ψ_2 is G_2-equivariant, so is μ_2, and thus μ_2 is a moment map for the G_2 action on $\mu_2^{-1}(0)$.

For the second part, assume that G acts freely on $\psi^{-1}(0)$. Suppose that $x \in \mu_2^{-1}(0)$, and that $g_2 \cdot x = x$ for some $g_2 \in G_2$. Let $\tilde{x} \in p_1^{-1}(x) \subset Z_1$. Then $g_2 \cdot \tilde{x} \in G_1 \cdot \tilde{x}$. This implies that for some $g_1 \in G_1$, $g_2 g_1^{-1} \cdot \tilde{x} = \tilde{x}$. But $\psi_1(\tilde{x}) = 0$ as $\tilde{x} \in Z_1$, and $\psi_2(\tilde{x}) = 0$ as $x \in \mu_2^{-1}(0)$. Hence $\tilde{x} \in \psi^{-1}(0)$, and so as G acts freely on $\psi^{-1}(0)$, $g_2 = g_1$. But as $G_1 \cap G_2 = \{1\}$, we get that $g_2 = 1$. Thus G_2 acts freely on $\mu_2^{-1}(0)$.

Finally, $\mu_2^{-1}(0) = p_1\left((\psi_2 \circ \iota_1)^{-1}(0)\right) = p_1\left(\psi_1^{-1}(0) \cap \psi_2^{-1}(0)\right) = p_1\left(\psi^{-1}(0)\right) = \psi^{-1}(0)/G_1$. Hence $\mu_2^{-1}(0)/G_2 = \left(\psi^{-1}(0)/G_1\right)/G_2 = \psi^{-1}(0)/G$. As the symplectic forms on both sides are induced from ω, they are equal, so the two reduced spaces are indeed symplectomorphic. □

2.3 The Delzant construction

Here we will discuss Delzant's theorem, which states that toric manifolds are classified by their moment polytopes. Let us start by introducing some vocabulary to describe convex polytopes.

Definition 2.3.1. A convex polytope $\Delta \subset \mathbb{R}^n$ is called **rational** if its facets (codimension-1 faces) have integral normal vectors, i.e. if it can be written as

$$\Delta = \{x \in \mathbb{R}^n : \langle x, \nu_i \rangle \geq -\kappa_i, 1 \leq i \leq d\}$$

for some $d > n$, $\kappa_i \in \mathbb{R}$, primitive integral vectors $\nu_i \in \mathbb{Z}^n$. A facet F_i of Δ can then be written as the set $\{x \in \mathbb{R}^n : \langle x, \nu_j \rangle \geq -\kappa_j \text{ for } j \neq i, \langle x, \nu_i \rangle = -\kappa_i\}$. Then ν_i is the primitive interior normal vector to F_i, and the κ_i are called **structure constants**.

Say n integral vectors in R^n are an **integral basis** if they are a $\mathbb{Z}$-basis for the lattice $\mathbb{Z}^n$.

Δ is **simple** if at each vertex there are n facets. It is **smooth** if in addition at each vertex the n primitive normal vectors to the facets meeting at that vertex form an integral basis. A **Delzant polytope** is a rational, simple, smooth polytope.

Δ is a **lattice polytope** (also called **integral**) if its vertices lie on the integer lattice $\mathbb{Z}^n$.

Example 2.3.2. For the moment polytope in Figure 2, the primitive interior normal vectors are $(1,0), (0,1)$ and $(-1,-1)$. The corresponding structure constants are 0,0,1. The polytope is thus rational, also simple and smooth, hence it is a Delzant polytope. It is also a lattice polytope. ◊

We will assume without proof he following fact:

Fact 2.3.3. *Smooth symplectic toric manifolds have smooth moment polytopes.*

Remark 2.3.4. Symplectic toric manifolds are also toric algebraic varieties, which have fans dual to the moment polytopes. The above fact comes from the fact that a toric algebraic variety X_Σ with fan Σ is smooth if and only if the fan Σ is smooth. See e.g. [7]. ◊

A foundational theorem proved by Thomas Delzant says that Delzant polytopes classify symplectic toric manifolds. Here we consider Delzant polytopes up to affine integral transformations, meaning up to translations and the action of $\mathrm{SL}_n(\mathbb{Z})$, and symplectic toric manifolds up to $\mathbb{T}^n$-equivariant symplectomorphisms.

Theorem 2.3.5. (Delzant) *There is a bijective correspondence between Delzant polytopes in $\mathbb{R}^n$ up to affine integral transformations and $2n$-dimensional symplectic toric manifolds up to equivariant symplectomorphisms.*

Proof. In one direction the map is just $M \mapsto \mu(M)$. Equivariantly symplectomorphic manifolds are related by a change of coordinates on the torus, which corresponds to an $\mathrm{SL}_n(\mathbb{Z})$ transformation of Δ. Different possible moment maps for a given M differ by a constant, so their moment polytopes differ by a translation. Hence this map is well-defined on our equivalence classes.

The reverse map is given by the Delzant construction, which given a Delzant polytope Δ constructs a symplectic toric manifold M_Δ. We will only prove that this reverse map is a left inverse to the map $M \mapsto \mu(M)$ (namely that the moment polytope of M_Δ is Δ). Proving that it is a right inverse (i.e. that a moment polytope determines a toric manifold uniquely up to equivariant symplectomorphism) is harder, a full proof can be found for example in [5].

Given a Delzant polytope Δ we shall construct a symplectic toric manifold M_Δ such that Δ is its moment polytope. M_Δ will be constructed as a symplectic reduction of $\mathbb{C}^d$ where d is the number of facets of Δ. Knowing Δ gives us two sets of information: the normal vectors ν_i and the structure constants κ_i. We also require two pieces of data to determine a symplectic reduction of $\mathbb{C}^d$: a torus acting on it, and a moment map for the action. The moment map is set up to a constant, and this constant will be determined by the κ_i, while the torus will come from the normal vectors ν_i. This will give us a bijective correspondence.

Suppose $\Delta = \{x \in \mathbb{R}^n : \langle x, \nu_i \rangle \geq -\kappa_i \text{ for } 1 \leq i \leq d\}$. Take the standard basis $\{e_i\}$ for $\mathbb{R}^d$ and let $\pi : \mathbb{R}^d \to \mathbb{R}^n$ be the map sending e_i to ν_i. Since Δ is smooth, a subset of the ν_i corresponding to the facets at a vertex is a basis for $\mathbb{Z}^n$, hence π is surjective. As the ν_i are integral, π maps the integer lattice $\mathbb{Z}^d$ onto $\mathbb{Z}^n$. Hence π descends to a map of tori $\hat{\pi} : \mathbb{T}^d \to \mathbb{T}^n$. Let $N = \ker \hat{\pi}$ a subtorus, $\mathfrak{n}$ its Lie algebra.

Thus we have the following exact sequences:

$$\begin{array}{lccccccccc}
\text{Tori:} & 0 & \longrightarrow & N & \hookrightarrow & \mathbb{T}^d & \xrightarrow{\hat{\pi}} & \mathbb{T}^n & \longrightarrow & 0 \\
\text{Lie algebras:} & 0 & \longrightarrow & \mathfrak{n} & \xhookrightarrow{\iota} & \mathbb{R}^d & \xrightarrow{\pi} & \mathbb{R}^n & \longrightarrow & 0 \\
 & & & & & e_i & \longmapsto & \nu_i & & \\
\text{Duals:} & 0 & \longrightarrow & (\mathbb{R}^n)^* & \xhookrightarrow{\pi^*} & (\mathbb{R}^d)^* & \xrightarrow{\iota^*} & \mathfrak{n}^* & \longrightarrow & 0
\end{array}$$

Now consider the standard $\mathbb{T}^d$ action on $\mathbb{C}^d$, with the moment map

$$\psi : (z_1, \ldots, z_n) \mapsto \left(|z_1|^2, \ldots, |z_d|^2\right) - (\kappa_1, \ldots, \kappa_d).$$

The image of ψ is $\{y \in (\mathbb{R}^d)^* : \langle y, e_i \rangle \geq -\kappa_i,\ i = 1, \ldots, d\}$.
The action restricts to an N action, which has moment map $\psi_N = \iota^* \circ \psi : \mathbb{C}^d \to \mathfrak{n}^*$.

Adding to our exact sequence:

$$\begin{array}{ccccccccc}
 & & & & \mathbb{C}^d & & & & \\
 & & & & \downarrow \psi & \searrow^{\psi_N} & & & \\
0 & \longrightarrow & (\mathbb{R}^n)^* & \xhookrightarrow{\pi^*} & (\mathbb{R}^d)^* & \xrightarrow{\iota^*} & \mathfrak{n}^* & \longrightarrow & 0
\end{array}$$

Let $Z := \psi_N^{-1}(0)$. We will show that N acts freely on Z and Z is compact, so we can define the reduced space $M_\Delta := Z/N$.

We need to show three things:

(i) Z is compact,

(ii) N acts feely on Z,

(iii) M_Δ has moment polytope Δ.

<u>(i) Z is compact</u>

Let $\Delta' = \pi^*(\Delta) \subset (\mathbb{R}^d)^*$. We will show that $Z = \psi^{-1}(\Delta')$. Since ψ is a proper map (as pre-images of bounded sets in $(\mathbb{R}^d)^*$ are bounded in $\mathbb{C}^d$), this will imply that Z is compact.

Indeed,

$$\begin{aligned}
Z &= \psi^{-1}\left(\ker \iota^*\right) = \psi^{-1}\left(\text{im } \pi^*\right) = \psi^{-1}\left(\text{im } \pi^* \cap \psi(\mathbb{C}^d)\right) \\
&= \psi^{-1}\left(\{y \in (\mathbb{R}^d)^* : \langle y, e_i\rangle \geq -\kappa_i,\ i = 1, \ldots, d;\ y = \pi^*(x) \text{ some } x \in (\mathbb{R}^n)^*\}\right) \\
&= \psi^{-1}\left(\{\pi^*(x) : x \in (\mathbb{R}^n)^*, \langle \pi^*(x), e_i\rangle \geq -\kappa_i,\ i = 1, \ldots, d\}\right) \\
&= \psi^{-1}\left(\{\pi^*(x) : x \in (\mathbb{R}^n)^*, \langle x, \pi(e_i)\rangle \geq -\kappa_i,\ i = 1, \ldots, d\}\right) \\
&= \psi^{-1}\left(\{\pi^*(x) : x \in (\mathbb{R}^n)^*, x \in \Delta\}\right) = \psi^{-1}\left(\pi^*(\Delta)\right) = \psi^{-1}\left(\Delta'\right).
\end{aligned}$$

Thus Z is compact.

(ii) N acts freely on Z

Denote the faces of Δ' by F'_I with $I \subset \{1, \ldots, d\}$, so that $F'_I = \{y \in \Delta' : \langle y, e_i\rangle = -\kappa_i \text{ for } i \in I\}$. Then $\psi^{-1}(F_I) = \{z \in Z : z_i = 0 \text{ for } i \in I\}$.

For $z \in Z$, since $Z = \psi^{-1}(\Delta')$, z lies in the preimage of the interior of some face $\psi^{-1}(\mathring{F}'_I)$.
Then $z_i = 0 \iff i \in I$, so the stabiliser of z under the $\mathbb{T}^d$ action is $\{t \in \mathbb{T}^d : t_i = 0 \text{ for } i \notin I\}$.
By relabelling the facets, without loss of generality $I = \{1, \ldots, r\}$, with $r \leq n$. Again relabelling if needed, without loss of generality $P' = F'_{\{1,\ldots,n\}}$ is a vertex of F'_I. $P' = \pi^*(P)$ for the vertex P of Δ which is the meeting point of the facets with normal vectors $\nu_1, \ldots, \nu_n$.

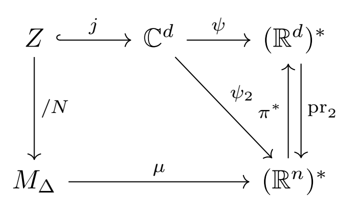
Let S be the stabiliser of $\psi^{-1}(P')$ under $\mathbb{T}^d$, so $S \geq \text{Stab}_{\mathbb{T}^d}(z)$. Let $\mathfrak{s}$ be the Lie algebra of S. Then $\mathfrak{s}$ is spanned by $e_1, \ldots, e_n$. Then $\pi(\mathfrak{s}) = \langle \nu_1, \ldots, \nu_n\rangle$, hence $\pi : \mathfrak{s} \to \mathbb{R}^n$ is bijective and sends the integer lattice to $\mathbb{Z}^n$ (as Δ is smooth). Hence $\hat{\pi} : S \to \mathbb{T}^n$ is a bijection.

As $N = \ker \hat{\pi}$, $N \cap S = \{1\}$ and so $N \cap \text{Stab}_{\mathbb{T}^d}(z) = \{1\}$. Hence N acts freely on z, and since z was any element of Z, N acts freely on Z.

(iii) M_Δ has moment polytope Δ

We saw above that taking a vertex of Δ gives us a bijection $\mathbb{T}^n \xrightarrow{\cong} S \leq \mathbb{T}^d$, and hence gives us a a splitting of the exact sequence $0 \longrightarrow N \hookrightarrow \mathbb{T}^d \xrightarrow{\hat{\pi}} \mathbb{T}^n \longrightarrow 0$, so we can write $\mathbb{T}^d = N \times \mathbb{T}^n$.

But now we are in the setup of Proposition 2.2.6 (reduction by stages). Let $j : Z \hookrightarrow \mathbb{C}^d$ be the inclusion, $\text{pr}_2 : (\mathbb{R}^d)^* \to (\mathbb{R}^n)^*$ projection onto the second factor, so $\psi_2 = \text{pr}_2 \circ \psi : \mathbb{C}^d \to (\mathbb{R}^n)^*$ is the moment map for the $\mathbb{T}^n$ action on $\mathbb{C}^d$.



Then as in Proposition 2.2.6, $\psi_2 \circ j$ descends to a map $\mu : M_\Delta \to (\mathbb{R}^n)^*$ which is a moment map for the $\mathbb{T}^n$ action on M_Δ. Now, $\text{im } \mu = \text{im } (\psi_2 \circ j) = \text{pr}_2(\psi(Z)) = \text{pr}_2(\pi^*(\Delta)) = \Delta$ since $\text{pr}_2 \circ \pi^*$ is the identity on $(\mathbb{R}^n)^*$. Thus indeed Δ is the moment polytope of M_Δ. □

As a corollary of the proof we obtain the following theorem:

Theorem 2.3.6. *For $x \in \Delta$, $\mu^{-1}(x)$ is a single torus orbit.*

Proof. Use the same notation as in the above proof. First not that for any $y \in \psi(\mathbb{C}^d)$, $\psi^{-1}(y)$ is a single $\mathbb{T}^d$ orbit.

Now let $x \in \Delta, y = \pi^*(x)$. Then

$$\begin{aligned}\mu^{-1}(x) &= \left((\psi_2 \circ j)^{-1}(x)\right)/N \\ &= \left((\psi \circ j)^{-1}(y)\right)/N \\ &= \left(\psi^{-1}(y)\right)/N \qquad \text{since } y \in \ker \iota^*\end{aligned}$$

Hence as $\psi^{-1}(y)$ is a single $\mathbb{T}^d$ orbit, $(\psi^{-1}(y))/N$ is a single $\mathbb{T}^n$ orbit. □

2.4 Some moment polytope geometry

Here we will give some auxiliary definitions and results that will simplify dealing with moment polytopes.

Definition 2.4.1. Suppose x is a vector with rational direction in $\mathbb{R}^n$. Then the **affine length** of x, denoted $l_{\text{aff}}(x)$ is the positive real λ such that $\frac{1}{\lambda}x$ is the smallest integer vector parallel to x.

Affine length is important because it is preserved by integral affine transformations, since they send primitive integral vectors to primitive integral vectors.

If Δ is a simple rational polytope in $(\mathbb{R}^n)^*$, at each vertex there are n edges coming out of it. An alternative characterisation of Δ being smooth is that the n primitive edge vectors form an integral basis.

Proposition 2.4.2. *A simple rational polytope in $(\mathbb{R}^n)^*$ is smooth if and only if for every vertex v of Δ, the n primitive integral edge vectors at v form an integral basis.*

Proof. Suppose the primitive integral normals at v are $\nu_1, \dots, \nu_n$ and the primitive integral edge vectors are $\epsilon_1, \dots, \epsilon_n$, where the facet normal to ν_i is spanned by $\epsilon_1, \dots, \hat{\epsilon}_i, \dots, \epsilon_n$. Thus $\forall i \neq j,\ \langle \epsilon_i, \nu_j \rangle = 0$.

Let N_v be the matrix with ith column ν_i, and E_v the matrix with ith column ϵ_i. We will show that $N_v \in \mathrm{SL}_n(\mathbb{Z}) \iff E_v \in \mathrm{SL}_n(\mathbb{Z})$.

Suppose that $N_v \in \mathrm{SL}_n(\mathbb{Z})$. We have $N_v^T \cdot \epsilon_i = (0, \dots, 0, \langle \nu_i, \epsilon_i \rangle, 0, \dots, 0)$. As N_v is an integer affine transformation, it preserves affine lengths, so $1 = l_{\text{aff}}(\epsilon_i) = l_{\text{aff}}\left((0, \dots, 0, \langle \nu_i, \epsilon_i \rangle, 0, \dots, 0)\right) = \langle \nu_i, e_i \rangle$.

So we see that $N_v^T \cdot E_v = I_n$. Since $N_v \in \mathrm{SL}_n(\mathbb{Z})$, also $E_v \in \mathrm{SL}_n(\mathbb{Z})$.

The proof in the other direction is exactly the same, and so we have proved that $N_v \in \mathrm{SL}_n(\mathbb{Z}) \iff E_v \in \mathrm{SL}_n(\mathbb{Z})$.

Since this holds for any vertex v of Δ, we are done. □

As a consequence of this result, given a smooth polytope with vertex x and edge vectors $\{e_i\}$ at x, we can by an integral affine transformation (equivalently by a change of coordinates) move x to any point of $(\mathbb{R}^n)^*$ and the edge vectors to any set $\{v_i\}$ which is an integral basis.

2.4.1 Almost global Darboux chart

We will need the following proposition for the method of probes: [1]

Proposition 2.4.3. *Let M be a $2n$-dimensional symplectic toric manifold, with moment polytope Δ and moment map $\mu : M \to \Delta$. Let x be an interior point of an edge ϵ of Δ. Let $\phi : \mathbb{C}^n \to P$ be a moment map for $\mathbb{C}^n$ such that x lies in the interior of an edge of P and further that P and Δ locally agree at x, in the sense that there is a Euclidean neighbourhood V of x such that $\Delta \cap V = P \cap V$.*

Let $\Delta' = \Delta \setminus \bigcup_{\substack{\text{facets } F \\ \text{with } x \notin F}} F$ be Δ with all facets not containing x removed, and similarly define

$P' = P \setminus \bigcup_{\substack{\text{facets } G \\ \text{with } x \notin G}} G$.

Then there is an equivariant symplectomorphism $g : \phi^{-1}(\Delta' \cap P') \to \mu^{-1}(\Delta' \cap P')$ and further $\mu \circ g = \phi$. Thus g gives an equivariant Darboux chart on $\mu^{-1}(\Delta' \cap P')$.

Proof. We can without loss of generality assume that $x = (0, \ldots, 0, 1)$ and that the interior normal vectors to the facets of Δ containing ϵ are $\nu_1 = e_1, \ldots, \nu_{n-1} = e_{n-1}$, a subset of the standard basis for $\mathbb{R}^n$. Assume also that $P = \{x_j \geq 0 \text{ for } j = 1, \ldots, n\}$.

Suppose Δ has d facets, with remaining normals $\nu_n = (a_{10}, a_{20}, \ldots, a_{(n-1)0}, 1)$, $\nu_{n+i} = (a_{1i}, a_{2i}, \ldots, a_{ni})$ for $i = 1, \ldots d - n$, $a_{ij} \in \mathbb{Z}$. Also the structure constants κ_i satisfy $\kappa_j = 0$ for $j = 1, \ldots, n-1$.

Then (using the notation of Theorem 2.3.5), $\pi : \mathbb{R}^d \to \mathbb{R}^n$ is given by the matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 & a_{10} & a_{11} & a_{12} & \ldots & a_{1\,d-n} \\ 0 & 1 & & \vdots & a_{20} & a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & & & \vdots & & \ddots & \\ & & & 1 & a_{n-1\,0} & a_{n-1\,1} & \ldots & & a_{n-1\,d-n} \\ 0 & \ldots & & 0 & 1 & a_{n1} & \ldots & & a_{n\,d-n} \end{pmatrix}$$

So $\mathfrak{n} = \ker \pi$ is spanned by the vectors

$$(a_{ni}\, a_{10} - a_{1i},\ a_{ni}\, a_{20} - a_{2i},\ \ldots,\ a_{ni}\, a_{n-1\,0} - a_{n-1\,i},\ -a_{ni},\ 0,\ \ldots,\ 1,\ \ldots,\ 0)$$

with the 1 in position $n + i$, for $i = 1, \ldots, d - n$.

Now let $\psi : \mathbb{C}^d \to (\mathbb{R}^d)^*$ be the moment map

$$\psi(z) = \left(|z_1|^2, \ldots, |z_n|^2 - \kappa_n, |z_{n+1}|^2 - \kappa_{n+1}, \ldots, |z_d|^2 - \kappa_d\right).$$

We know from the Delzant construction that $M = Z/N$, where $Z = (\iota^* \circ \psi)^{-1}(0)$, and N the subtorus of $\mathbb{T}^d$ with Lie algebra $\mathfrak{n}$. Now,

$$\begin{aligned} (\iota^*)^{-1}(0) &= \{y \in (\mathbb{R}^d)^* : \iota^* y = 0\} \\ &= \{y \in (\mathbb{R}^d)^* : y(\iota(v)) = 0\ \forall v \in \mathfrak{n}\} \\ &= \left\{ y \in (\mathbb{R}^d)^* : \text{ for } i = 1, \ldots, d-n,\ y_{n+i} = \sum_{j=1}^{n} a_{ji}\, y_j - \sum_{j=1}^{n-1} a_{j0}\, a_{ni}\, y_j \right\}. \end{aligned}$$

[1] The author would like to thank Jack Smith for showing him this proof.

Hence

$$Z = \left\{ z \in \mathbb{C}^d : \text{ for } i = 1, \ldots, d-n, \ |z_{n+i}|^2 - \kappa_{n+i} = \sum_{j=1}^{n-1} (a_{ji} - a_{j0}\, a_{ni})|z_j|^2 + a_{ni}(|z_n|^2 - \kappa_n) \right\}.$$

If $Z' = Z \cap \{z \in \mathbb{C}^d : |z_{n+i}| > 0 \ \text{ for } i = 0, \ldots, d-n\}$, then $\mu^{-1}(\Delta' \cap P') = Z'/N$.

But if we take

$$\mathbb{C}^n \supset U := \left\{ z \in \mathbb{C}^n : \text{ for } i = 1, \ldots, d-n, \ \sum_{j=1}^{n-1} (a_{ji} - a_{j0}\, a_{ni})|z_j|^2 + a_{ni}(|z_n|^2 - \kappa_n) + \kappa_{n+i} > 0 \right\} = \phi^{-1}(\Delta' \cap P'),$$

then we can define the map

$$\begin{aligned} \tilde{F} : U &\longrightarrow Z' \\ (z_1, \ldots, z_n) &\longmapsto (z_1, .., z_n, w_1, \ldots, w_{d-n}), \quad \text{where } w_i = \sqrt{\textstyle\sum_{j=1}^{n-1} (a_{ji} - a_{j0}\, a_{ni})|z_j|^2 + a_{ni}(|z_n|^2 - \kappa_n) + \kappa_{n+i}}. \end{aligned}$$

Since each orbit of N on $\mathbb{C}^d$ has a unique representative with all of $z_{n+1}, \ldots, z_d$ real, $\tilde{F}$ descends to a bijective map $F : U \to Z'/N$, thus a diffeomorphism between $U = \phi^{-1}(\Delta' \cap P')$ and $\mu^{-1}(\Delta' \cap P')$.

F is a symplectomorphism: if ω_d, ω_n are the standard symplectic forms on $\mathbb{C}^d$, $\mathbb{C}^n$ respectively, then $\tilde{F}^*\omega_d = \tilde{F}^*(\sum \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j + \sum \mathrm{d}w_j \wedge \mathrm{d}\bar{w}_j)$, but as the w_i are always real in the image of $\tilde{F}$, the second part vanishes, and so $\tilde{F}^*\omega_d = \omega_n$. Hence $F^*\omega = \omega_n$ by construction of ω.

The symplectomorphism is equivariant as the action on M descends from the standard torus action on $\mathbb{C}^d$, which restricts to the torus action on $\mathbb{C}^n$.

Finally, from the uniqueness of moment maps, since ϕ and $\mu \circ g$ are both moment maps for the same $\mathbb{T}^n$ action on U, they can only differ by a constant. Since they have the same image, they must be equal, so $\phi = \mu \circ g$. □

Remark 2.4.4. The above proposition is one case of a slightly more general statement, where x can be the interior point of a face f of any dimension. If f has higher dimension, the condition of the proposition is weaker since we need $n - \dim f$ facets of Δ to lie in the same hyperplanes as facets of P. However the implication is also weaker as Δ' contains less facets of Δ so we get the Darboux chart on a smaller neighbourhood. ♢

2.4.2 Sections of moment polytopes by planes

We noted in Remark 2.2.4 that if M_{red} is a symplectic reduction of M, and they have moment polytopes $\Delta_{\text{red}}, \Delta$ respectively, then Δ_{red} can be obtained from Δ by taking the intersection with a linear subspace of $(\mathbb{R}^n)^*$.

We can also go the other way. Given a moment polytope $\Delta \subset (\mathbb{R}^n)^* = \mu(M)$ of M, we can consider $\Delta' = \Delta \cap L$ where L is an integral r-dimensional linear subspace of $(\mathbb{R}^n)^*$ (by integral we mean that is is spanned by a subset of an integral basis of $(\mathbb{R}^n)^*$). Since L is integral, $(\mathbb{R}^n)^*/L$ generates a torus $K \leq \mathbb{T}^n$, and provided 0 is a regular value of μ_K and K acts freely on $\mu_K^{-1}(0)$, Δ' is the moment polytope of the symplectic reduction of M relative to the torus K.

Now, if 0 is an interior point of Δ, then it will automatically be a regular value of μ_K. Let us determine when K acts freely on $\mu_K^{-1}(0)$.

$\mathbb{T}^n$ acts freely on $\mu^{-1}(\mathring{\Delta})$, so issues can only arise on the boundary. If $L^\perp = \text{span}\{\eta_1, \ldots, \eta_r\}$, then $\text{Lie}(K)$ is generated by $\eta_1, \ldots \eta_r$. Consider a point on the intersection of L with the interior of the face f, where $f = \cap_{i \in I} F_i$. The stabiliser of points in $\mu^{-1}(\mathring{F}_i)$ is the torus generated by the normal ν_i, so points in $\mathring{f}$ are stabilised by the torus generated by $\{\nu_i : i \in I\} = f^\perp$. Thus K acts freely on $\mu_K^{-1}(0)$ if and only for every intersection of L with $\mathring{f}$ for any face f, $L^\perp \cap f^\perp = \{0\}$.

Since L and f always meet transversally (in the real sense), if codim $L +$ codim $f \leq n$, then the intersection is trivial, otherwise it is not. Thus the condition for L to give a valid symplectic reduction of M is that M should not meet any faces of dimension lower than codim L.

To summarise:

Proposition 2.4.5. *Let M a toric manifold with a moment polytope $\Delta \subset (\mathbb{R}^n)^*$ such that $0 \in \mathring{\Delta}$. Let $L \subset (\mathbb{R}^n)^*$ an integral linear subspace. Then the section of Δ by L, $\Delta' = \Delta \cap L$, gives a symplectic reduction of M provided that L does not intersect any faces of Δ of dimension lower than codim L.*

2.5 Monotone toric manifolds

Monotone symplectic manifolds are a special particularly nice class of symplectic manifolds for which the symplectic structure is naturally related to their topology. Monotone toric manifolds have particularly simple structure, which allows us to prove much stronger displaceability results in this setting (sections 3.2 and 4.2).

Definition 2.5.1. A compact symplectic manifold (M, ω) is **monotone** if ω is a multiple of its first Chern class [2], i.e. $[c_1(M)] = \lambda[\omega]$ for some $\lambda \in \mathbb{C}$.

For us monotone symplectic toric manifolds will be characterised by the following fact:

Fact 2.5.2. *If Δ is a Delzant polytope with all the structure constants κ_i equal, then the corresponding toric manifold M_Δ is monotone. Conversely if M is monotone, its moment polytope Δ can be translated so that all the structure constants are equal.*

Proof. (Sketch)
We will take the following fact for granted. Let M have moment polytope $\Delta = \{x : \langle x, \nu_i \rangle \geq -\kappa_i\}$. Seeing M as a toric variety, it has toric divisors D_i corresponding to the normals ν_i. Then:

Fact 2.5.3.

$$[c_1(M)] = \sum_i PD(D_i), \quad [\omega] = \sum_i \kappa_i PD(D_i)$$

From this we immediately see that if Δ has all structure constants equal then M is monotone.

Conversely suppose that M is monotone. A fact about toric divisors (see e.g. [7]) is that all the relations between them are linear combinations of the relations

$$\sum_i (\nu_i)_j D_i = 0 \quad \text{for } j = 1, \ldots, n.$$

As M is monotone, there exists a λ such that $[\omega] = \lambda[c_1(M)]$, so $\sum_i (\kappa_i - \lambda) D_i = 0$.

That means that for some a_j, $\kappa_i - \lambda = \sum_j a_j (\nu_i)_j$ for any i. Hence for any i, $\langle a, \nu_i \rangle - \kappa_i = -\lambda$.

The translate of Δ by $a = (a_1, .., a_n)$ is

$$\{x : \langle x - a, \nu_i \rangle \geq -\kappa_i\} = \{x : \langle x, \nu_i \rangle \geq \langle a, \nu_i \rangle - \kappa_i\} = \{x : \langle x, \nu_i \rangle \geq -\lambda\}.$$

So $\Delta + a$ has all structure constants equal. ∎

[2] All symplectic toric manifolds are Kähler (see e.g. [1]), so it makes sense to talk about their first Chern class.

From now on we will assume that for monotone toric manifolds the symplectic form is scaled such that $[\omega] = [c_1(M)]$, so the structure constants are all 1.

The moment polytopes corresponding to monotone manifolds are called monotone:

Definition 2.5.4. A smooth polytope in $(\mathbb{R}^n)^*$ is **monotone** if all the structure constants are equal to 1.

Let Δ be a monotone polytope. Then $\Delta = \{x : \langle x, \nu_i \rangle \geq -1\}$ for some normals ν_i.
Take x a vertex of Δ, by relabelling say the n facets meeting at x are normal to $\nu_1, \ldots, \nu_n$ respectively. As Δ is smooth, these normals are an integral basis. Since $\langle x, \nu_i \rangle = -1 \in \mathbb{Z}$ for all i, this implies that $\langle x, e_i \rangle \in \mathbb{Z}$ for all i, where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$. Hence $x \in \mathbb{Z}^n$ so we see that Δ is a lattice polytope.

As $0 > -1$, we see that 0 is an interior lattice point of Δ. Suppose y is another interior lattice point. Then the ray from 0 through y meets some face F_i. Then $\langle y, \nu_i \rangle < 0$. But as y is a lattice point, $\langle y, \nu_i \rangle$ must be an integer, and it must be greater than -1 as y is an interior point of Δ. This is a contradiction, so such a y can not exist.

Thus we have proved:

Proposition 2.5.5. *Monotone polytopes are integral and have a unique interior lattice point, which is* 0.

There is an alternate characterisation of moment polytopes in terms of edge vectors, analogously to the alternate characterisation of smoothness in terms of edge vectors.

Proposition 2.5.6. *A smooth polytope is monotone if and only if it satisfies the vertex-Fano condition: at each vertex x, if $\epsilon_1, .., \epsilon_n$ are the n primitive edge vectors at x, then*

$$x + \sum_{i=0}^{n} \epsilon_i = 0.$$

Proof. Let x a vertex of Δ, let interior primitive integral normals at x be ν_i and primitive edge vectors ϵ_i. As shown in the proof of Proposition 2.4.2, if N_x is the matrix with ith column ν_i and E_x the matrix with ith column ϵ_i, then $N_x^T E_x = I$.

Now for any vertex x,

$$N_x^T x = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} \iff x = E_x \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} = -\sum_i \epsilon_i$$

The left condition for all x is equivalent to Δ being monotone, the right condition for all x is the vertex-Fano condition. Thus we have shown that they are equivalent. □

3 Showing displaceability – the method of probes

With the prerequisites set up, we can finally get to our first method of determining displaceability. We will first introduce the method of probes, then give a few examples of applications. It turns out that in the case of monotone polytopes the displaceability of all fibres except the special central fibre can be deduced just from a combinatorial condition on the moment polytope, the star Ewald condition. However the method is not always this strong and we will give an example with an open set of fibres that can not be displaced by probes. This section follows [14], though some proofs are slightly altered.

3.1 The method of probes

Definition 3.1.1. An integral vector u in $\mathbb{R}^n$ is said to be **integrally transverse** to a rational hyperplane H, if there is a set of vectors $v_1, \ldots, v_{n-1}$ parallel to H such that $\{u, v_1, \ldots, v_{n-1}\}$ is an integral basis.

Let $w \in \mathring{F}$ for some facet F of a rational polytope $\Delta \subset (\mathbb{R}^n)^*$, let $\lambda \in \mathbb{Z}^n$ be integrally transverse to F. The **probe** $p_{F,\lambda}(w)$ at point w in the direction λ is the intersection of $\mathring{\Delta}$ with the ray from w in the direction λ.

Theorem 3.1.2. (The method of probes)*[14, Lemma 2.4]*
Suppose M is a compact smooth symplectic toric manifold, with moment polytope $\Delta = \mu(M)$. Suppose F is a facet of Δ, $p = p_{F,\lambda}(w)$ a probe from a point $w \in \mathring{F}$. Then for points x lying on the probe p, less than halfway along p from w, the fibre $\mu^{-1}(x)$ is displaceable.

Proof. Without loss of generality we may assume that Δ lies in the positive orthant, $F \subset \{x_1 = 0\}$, $\lambda = (1, 0, \ldots, 0)$, an edge ϵ of F lies on the x_n-axis and the other facets containing this edge lie in the hyperplanes $\{x_i = 0\}$ for $i = 2, \ldots, n-1$. Hence we can use Proposition 2.4.3 applied to any interior point of ϵ to see that there is a global Darboux chart on $\mu^{-1}(\Delta')$, for $\Delta' = \Delta \setminus \bigcup_{\substack{\text{facets } F\\ \text{with } \epsilon \not\subset F}} F$.

If a is the affine length of the probe p, in this chart $\mu^{-1}(p)$ is:

$$\mu^{-1}(p) = \{z \in \mathbb{C}^n : |z_1|^2 < a, |z_i| = w_i\}, \text{ where } w = (0, w_2, \ldots, w_n = 1).$$

Note that since w lies in the interior of F, the w_i are positive for $i \geq 2$.

Hence, if we denote by $D^2(A)$ the disc in $\mathbb{C}$ with centre 0 and area πA, we have that $\mu^{-1}(p)$ is symplectomorphic to $D^2(a) \times \mathbb{T}^{n-1}$ with the symplectic form the area form on the $D^2(a)$ factor.
For $x \in p$ this symplectomorphism takes $\mu^{-1}(x)$ to $\partial D^2(b) \times \mathbb{T}^{n-1}$, where $b = l_{\text{aff}}(x - w)$.

But if $b < \frac{a}{2}$, the circle $\partial D^2(b)$ can be displaced in $D^2(a)$ by an area-preserving isotopy (see Figure 3). Let V be the vector field on $D^2(a)$ generating the isotopy. As the isotopy is area-preserving, V is a symplectic vector field. Then if ω_1 is the symplectic form on $\mathbb{C}$, which restricts to the symplectic form on $D^2(a)$, $\iota_V \omega_1$ is closed, and hence exact as $H^2(D^2) = 0$. So $\iota_V \omega_1 = \mathrm{d}H$ for some $H : D^2(a) \to \mathbb{R}$. We can extend H to $\tilde{H} : D^2(a) \times \mathbb{T}^{n-1} \to \mathbb{R}$, keeping it constant along the torus factor, and then extend it to M using a bump function, so that $\tilde{H} : M \to \mathbb{R}$ has compact support. Then $\tilde{H}$ generates a Hamiltonian vector field that displaces $\partial D^2(b) \times \mathbb{T}^{n-1}$ within $D^2(a) \times \mathbb{T}^{n-1}$, and hence the fibre $\mu^{-1}(x)$ is displaceable in M. □

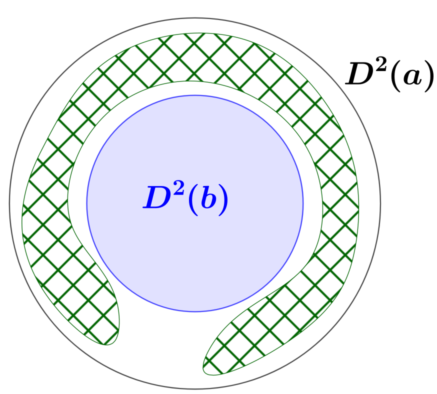


Figure 3: If $b < \frac{a}{2}$, $D^2(b)$ can be displaced within $D^2(a)$ by an area-preserving isotopy, by just 'pushing' the small disc out towards the boundary of the large disc.

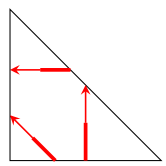
Definition 3.1.3. If Δ is a rational polytope, $x \in \mathring{\Delta}$, we say that x is **displaceable by probe** if there exists a probe $p_{F,\lambda}(w)$ from some interior point w of a facet F of Δ such that x is less than halfway along $p_{F,\lambda}(w)$.

Example 3.1.4. For Δ the unit interval, the probe p_{0,e_1} displaces all points $x < \frac{1}{2}$, and the probe $p_{1,-e_1}$ displaces all points $x > \frac{1}{2}$. The point $x = \frac{1}{2}$ is non-displaceable by probe, and we already know that the corresponding fibre is non-displaceable. $\Diamond$

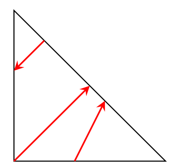
Remark 3.1.5. We can see from the proof why it is important that p is less than halfway along the probe, since the circle $\partial D^2(b)$ is not displaceable in $D^2(a)$ for $b \geq \frac{a}{2}$. We can also see why the conditions in the definition of probes were necessary. The direction needs to be integrally transverse to the face so that we can do the change of coordinates that we need, as $(1, 0, \ldots, 0)$ is integrally transverse to $\{x_1 = 0\}$.

We also need w to lie in the interior of F: if w lies in the interior of a face of dimension k, then $\mu^{-1}(w) \cong \mathbb{T}^k$. $\mu^{-1}(p)$ is diffeomorphic to $D^2 \times \mathbb{T}^{n-1}$ if and only if $\mu^{-1}(w)$ is diffeomorphic to $\{0\} \times \mathbb{T}^{n-1}$, thus if and only if $k = n - 1$. Hence we need w to lie in the interior of a facet.

We give some examples and non-examples of valid probes in Figure 4.



(a) A few valid probes for $\mathbb{C}\mathrm{P}^2$. The arrows denote the directions of the probes, the fat part of the probe denotes the points less than halfway along it.



(b) These are not valid probes, they are either not integrally transverse to their facet or do not originate at an interior point of a facet.

Figure 4: Examples and non-examples of probes in 2 dimensions.

$\Diamond$

3.2 Monotone polytopes

In [14] McDuff introduces a condition on monotone polytopes, called the star Ewald condition, which she shows is equivalent to every interior point except the central point being displaceable by probe. A computer check by Paffenholz [15] showed that all monotone polytopes in dimension ≤ 5 satisfy this condition, but for higher dimensions there are counterexamples. Hence for all monotone toric manifolds in dimension ≤ 10 all toric fibres except the central one are displaceable. As we shall see in section 4, the method of Abreu and Macarini shows that the central fibre is never displaceable, hence for monotone toric manifolds in dimensions ≤ 10 we get a complete characterisation of displaceable fibres.

Here we give the proof that the Star Ewald condition implies the displaceability of all but the central point, and check that all 2– and 3–dimensional monotone polytopes satisfy it.

3.2.1 Star Ewald condition

Definition 3.2.1. [14, Definition 3.5]
Let Δ a smooth polytope in $\mathbb{R}^n$ containing 0 in its interior. Let F_i be its facets. Denote by $\mathcal{S}(\boldsymbol{\Delta})$ the set of integral symmetric points of Δ, i.e. $\mathcal{S}(\Delta) = \mathbb{Z}^n \cap \Delta \cap -\Delta$.

For a face $f = \bigcap_{i \in I} F_i$ of Δ, let $\mathbf{Star}(f) = \bigcup_{i \in I} F_i$ the union of facets containing f, and let $\mathbf{star}(f) = \bigcup_{\substack{i,j \in I \\ i \neq j}} F_i \cap F_j$ the union of codimension 2 faces containing f.

Then define the **deleted star** of f as $\mathrm{Star}^*(f) = \mathrm{Star}(f) \setminus \mathrm{star}(f)$.

We say a face f of Δ satisfies the **star Ewald condition** if there exists a $\lambda \in \mathcal{S}(\Delta) \cap \mathrm{Star}^*(f)$ such that $-\lambda \notin \mathrm{Star}(f)$.
Say Δ is **star Ewald** if all its faces satisfy the star Ewald condition.

Theorem 3.2.2. *[14, Theorem 1.2]*
If a monotone polytope Δ is star Ewald, then every point in $\mathring{\Delta} \setminus \{0\}$ is displaceable by probe.

Proof. For a face f of Δ and a point $x \in \Delta \setminus f$, write

$$C(f, x) := \{rx + (1-r)y : r \in (0,1), y \in \mathring{f}\}$$

for the relative interior of the cone on f with cone point x. (Here $\mathring{f}$ is the relative interior of f and by convention the interior of a vertex is the vertex itself).

Then $\mathring{\Delta} \setminus \{0\} = \bigcup_f C(f, 0)$, since for any $x \in \mathring{\Delta} \setminus \{0\}$, the ray from 0 through x crosses $\mathring{f}$ for some face f and so $x \in C(f, 0)$. We will show that if f is star Ewald, then every point in $C(f, 0)$ is displaceable by probe.

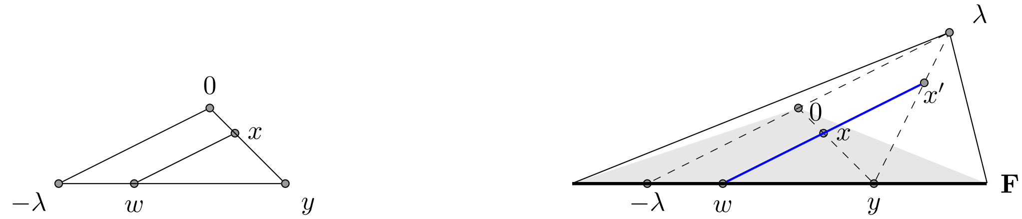
First consider the case where f is a facet F. Then $\mathrm{Star}(F) = \mathrm{Star}^*(F) = F$.
Suppose $-\lambda \in F \cap \mathcal{S}$. Then $\lambda \in \Delta \setminus F$ (as F does not contain 0, it can not contain symmetric points), so $C(F, \lambda) \subset \mathring{\Delta}$ by convexity. The vector λ is integrally transverse to F: without loss of generality $F = \{x_1 = -1\}$, then as $-\lambda \in F$, the first coordinate of λ is 1.

Then any point in $C(F, 0)$ is displaceable by a probe $p_{F,\lambda}(w)$ for some $w \in \mathring{F}$. Indeed, if $x \in C(F, 0)$, then $x = ry$ for some $y \in \mathring{F}, r \in (0,1)$. Now, $x - (1-r)\lambda = ry + (1-r)(-\lambda)$ lies on the open segment $(-\lambda, y) \subset \mathring{F}$ so $x = w + (1-r)\lambda$ for some $w \in \mathring{F}$, and $w + 2(1-r)\lambda = ry + (1-r)\lambda \in C(F, \lambda) \subset \mathring{\Delta}$. Hence x is displaceable by probe from w in direction λ. (See Figure 5 for illustration.)

Similarly for faces f of lower dimension:
Suppose $-\lambda \in \mathrm{Star}^*(f) \cap \mathcal{S}$ and $\lambda \notin \mathrm{Star}(f)$. Then $C(f, \lambda) \subset \mathring{\Delta}$, as otherwise $C(f, \lambda)$ lies in some facet F, but then $\lambda \in F$ and $f \subset F$, so $\lambda \in \mathrm{Star}(f)$.

Since $-\lambda$ is not in a codimension–2 face containing f, it must be in the interior of a facet F containing f, and so $-\lambda \notin f$. Hence we can define $W = C(f, -\lambda)$. Since $\lambda \notin F$, $C(W, \lambda) \subset \mathring{\Delta}$.

Now for $x \in C(f, 0)$, if $x = ry$ for $y \in \mathring{f}$, then $w = x - (1-r)\lambda$ lies in $\mathring{W} \subset \mathring{F}$, and $w + 2(1-r)\lambda = ry + (1-r)\lambda \in C(f, \lambda) \subset \mathring{\Delta}$. Hence x is displaceable by probe from w in direction λ. $\square$



(a) If $x = ry$, then $w = x - (1-r)\lambda$ lies on the segment between $-\lambda$ and y.

(b) The cone $C(F, 0)$ is shaded, the probe $p_{F,\lambda}(w)$ is shown in blue. As $x' = 2x - w$ lies on the segment between y and λ, x is less than halfway along the probe.

Figure 5: If $-\lambda \in \mathrm{Star}^*(F) \cap \mathcal{S}$ then any $x \in C(F, 0)$ is displaceable by probe with direction λ starting from some $w \in \mathring{F}$.

The only monotone 1–dimensional polytope is the segment $[-1, 1]$, and it is clear that all points other than the central point are displaceable by probe. The case $n = 2$ and $n = 3$ are more interesting but still simple to do by hand.

3.2.2 2–dimensional monotone polytopes

Lemma 3.2.3. *There are five 2–dimensional monotone polytopes, which are shown in Figure 6. They are the moment polytopes of* $\mathbb{C}P^2$ *with* 0, 1, 2 *or* 3 *blow-ups, and of* $\mathbb{C}P^1 \times \mathbb{C}P^1$.

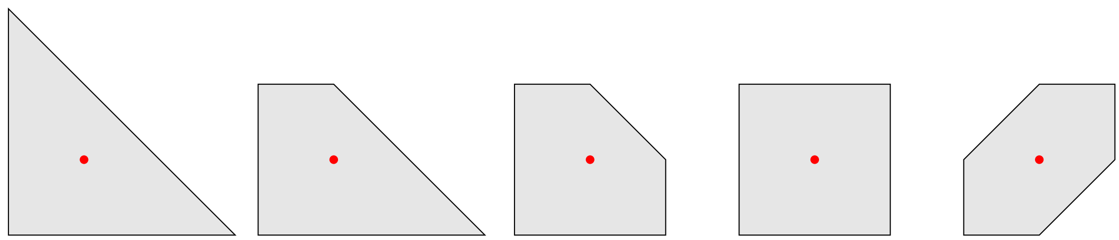


Figure 6: The five 2–dimensional monotone polytopes. The origin is marked in red on each, the lower left corner is $(-1, -1)$.

Proof. Without loss of generality we assume one of the vertices is at $(-1, -1)$ and the edges there are $\{x = -1\}$, $\{y = -1\}$. These edges can have length 1, 2, or 3, and by considering these cases, and remembering that $(0, 1)$ and $(1, 0)$ must be on the boundary for the polytope to be monotone, we see that there are five smooth possibilities. □

Proposition 3.2.4. *Every 2–dimensional monotone polytope is star Ewald.*

Proof. This is a trivial check. For example for the hexagon (the rightmost polytope in Figure 6), the following λ work:

- $\lambda = (-1, -1)$ for the edges $x = 1$, $y = 1$, and vertices $(1, 0)$ and $(0, 1)$.
- $\lambda = (1, 1)$ for the edges $x = -1$, $y = -1$ and vertices $(-1, 0)$ and $(0, -1)$.
- $\lambda = (-1, 0)$ for the edge $x - y = 1$ and vertex $(1, 1)$.
- $\lambda = (1, 0)$ for the edge $x - y = -1$ and vertex $(-1, -1)$.

□

3.2.3 3–dimensional monotone polytopes

In this section Δ is a monotone 3–dimensional polytope. As before we assume $(-1, -1, -1)$ is a vertex and the facets there are $F_i = \{x_i = -1\}$ for $i = 1, 2, 3$.

Definition 3.2.5. Say a face of Δ is **small** if it is a triangle with an edge of length 1.

Claim. *A small face has all edges of length 1.*

Proof. Without loss of generality the vertices are $(-1, -1, -1), (-1, 0, -1)$ and $(-1, -1, z)$ for some $z \in \mathbb{Z}$. But the vertex-Fano equation at $(-1, 0, -1)$ shows that $z = 0$, so we are done. □

Lemma 3.2.6. *[14, Lemma 4.8]*
If Δ has a small face, it is one of the two polytopes in Figure 7.

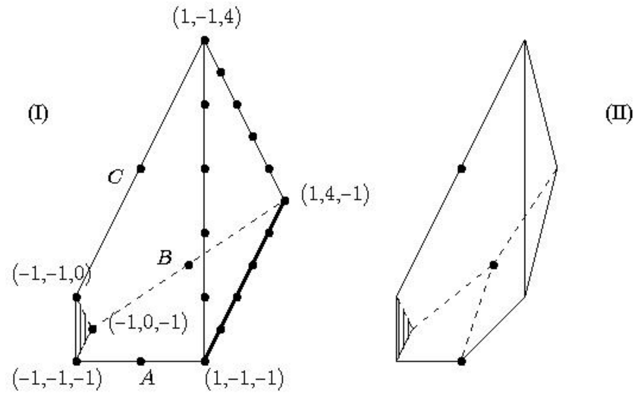


Figure 7: The two three–dimensional monotone polytopes with a small face (shaded). Lattice points on the edges are marked for (I). Figure taken from [14].

Proof. Without loss of generality the small face is F_1, with vertices $v_1 = (-1,-1,-1)$, $v_2 = (-1,0,-1)$, $v_3 = (-1,-1,0)$. Let the third primitive edge vectors at v_i be ϵ_i. Then the vertex-Fano equations give us: $\epsilon_1 = (1,0,0)$, $\epsilon_2 = (1,2,0)$, $\epsilon_3 = (1,0,2)$. Let $A = v_1 + \epsilon_1 = (0,-1,-1)$, $B = v_2 + \epsilon_2 = (0,2,-1)$, $C = v_3 + \epsilon_3 = (0,-1,2)$.

Case 1: None of A, B, C are vertices.

Let $A' = v_1 + 2\epsilon_1 = (1,-1,-1)$, $B' = v_2 + 2\epsilon_2 = (1,4,-1)$, $C' = v_3 + 2\epsilon_3 = (1,-1,4)$. All of A', B', C' lie in Δ, hence $(1,1,1)$ does as well by convexity. By monotonicity, it must be on a face, so A', B', C' must span a face and hence are vertices. Then Δ is the convex hull of $v_1, v_2, v_3, A', B', C'$, which is the polytope (I).

Case 2: Some of A, B, C are vertices.

Without loss of generality A is a vertex. The edge vectors at A are then $(-1,0,0)$ and some $f_2 \in F_2$ and $f_3 \in F_3$. Write $f_2 = (k,0,1)$, $f_3 = (l,1,0)$ with $k, l \in \mathbb{N}$. Then by the vertex-Fano equation, $k + l = 1$, and both must be non-negative (or else B, C respectively land outside Δ), so one is 1, the other 0. Without loss of generality $k = 1$, $l = 0$. Then $B = A + 3f_3$ so B is a vertex. Then C can't be a vertex or else 0 lands on the face ABC while it must be in the interior. So C' must be in Δ.

$D = A + f_2 = (1,-1,0)$ must be on the third edge from A, and by the vertex-Fano equation at B, the point $E = (1,3,0)$ is on the third edge from B. This tells us that $\Delta \cap \{-1 \le x_1 \le 1\}$ is spanned by v_1, v_2, v_3, C', D, E. But as $(1,0,1)$ lies in the interior of the span of $C'DE$, the points C', D and E must span a face and so $\Delta = \Delta \cap \{-1 \le x_1 \le 1\}$, which is the polytope (II). □

Corollary 3.2.7. *Δ can have at most one small facet, and if F is small, then the lattice points of F lie in $\mathcal{S}(\Delta)$.*

Proof. Both (I) and (II) have a single small facet, and we can see that $-F$ lies in Δ in both cases. □

We need a few more definitions to simplify the proof that all 3–dimensional monotone polytopes are star Ewald.

Definition 3.2.8. Suppose v is a vertex of facet F. Let ϵ_1, ϵ_2 be the two primitive edge vectors at v spanning F. Then the **special point** of v in F is $s_{v,F} = v + \epsilon_1 + \epsilon_2$.

For example, with notation as above, $s_{v_1,F_1} = (-1,0,0)$, $s_{v_1,F_2} = (0,-1,0)$, $s_{v_1,F_3} = (0,0,-1)$. Note that by the vertex-Fano condition, $s_{v,F} = -\epsilon$ where ϵ is the edge vector at v transverse to F.

Remark 3.2.9. Observe that if $s_{v,F} \in \Delta$, then $s_{v,F} \in \mathrm{Star}^*(v)$, and $-s_{v,F} \notin \mathrm{Star}(v)$. Hence if $s_{v,F} \in \mathcal{S}(\Delta)$, then v satisfies the star Ewald condition. Also note that unless F is small, $s_{v,F} \in F$. ◊

Definition 3.2.10. Suppose v is a vertex of facet F, and e is the edge at v transverse to F. Let the other end of e be the vertex w, and let F' be the facet at w not containing e. Then F' is called the **facet opposite to F at v**.

If $s = s_{v,F}$, call $s_{w,F'}$ the **opposite special point**, and denote it s'.

Observe that $s' = -s$ since they are $-\epsilon$ and ϵ respectively where ϵ is the primitive integral vector parallel to e.

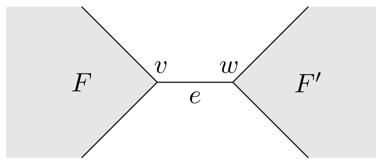


Figure 8: F' is opposite to F at v.

Proposition 3.2.11. *[14, Proposition 4.7]*
All monotone 3–dimensional polytopes are star Ewald.

Proof. We will check the star Ewald condition for the vertices, edges and faces of Δ.

Vertices
Take $v = (-1, -1, -1)$ as before. Let $s = s_{v,F_1}$ and let s' be the opposite special point. If neither F_1 nor F_1' are small, then both s and s' lie in Δ, hence $s \in \mathcal{S}(\Delta)$. Then by Remark 3.2.9, v satisfies the star Ewald condition.

Observe that for both polytopes (I) and (II), each vertex lies on a non-small facet such that the opposite facet is also not small. Hence all vertices satisfy the star Ewald condition.

Edges
If Δ has no small facets: without loss of generality e is the edge $e = F_2 \cap F_3$ parallel to $(1, 0, 0)$, and $v_1 = (-1, -1, -1)$ a vertex of e. Then if $s = s_{v,F_3} = (0, 0, -1)$, $s \in \mathcal{S}(\Delta)$ since there are no small facets. Since $s \notin e = \mathrm{star}(e)$, and $-s \notin F_2 \cup F_3 = \mathrm{Star}(e)$, $-s$ is the point λ needed to check that e satisfies the star Ewald condition.

If Δ does have a small facet: we can check that every edge e of (I) and (II) lies in a non-small facet F with an opposite non-small facet F' at v such that e is not an edge of F'. Then $\lambda = -s_{v,F}$ works again.

Faces
If F is small, then by Corollary 3.2.7, any vertex lies in $\mathcal{S}(\Delta)$ and so is the $-\lambda$ we need.

Otherwise, let F_w' be the opposite face at vertex w. If all F_w' are small, then by inspection F must be the large triangular face in (I), which contains $(1, 1, 1) \in \mathcal{S}(\Delta)$. If there is a non-small opposite face F_w', then $s_{w,F}$ lies in $\mathcal{S}(\Delta)$ and so we are done in this case too. □

Thus we have completed the proof of:

Theorem 3.2.12. *For monotone smooth polytopes Δ of dimension ≤ 3, all points in $\mathring{\Delta}$ except the central point are displaceable by probe.*

3.3 Example: open set of points non-displaceable by probe

Here we give an example of McDuff ([14, Lemma 4.4]) of a 2-dimensional smooth polytope with an open set of points that are not displaceable by probe.

Note that if a probe exits through a face that it is integrally transverse to, then all points of the probe except for the midpoint are displaceable. Hence for there to exist an open set of points non-displaceable by probe, it is necessary for some probes to exit through faces they are not integrally transverse to. This is the idea behind this example – to construct a polytope where most probes exit through faces they are not integrally transverse to.

The second idea, which simplifies the construction, is to start with a non-smooth polytope, and to smooth it out later by cutting the corners. Since these cuts can be arbitrarily small, and for each new edge only a finite set of directions of probe displaces new points, the set of new points displaceable by probes after the smoothening can be arbitrarily small.

We start with the triangle with vertices $A = (0, 5),\ B = (0, 0),\ C = (3, 0)$, and consider which points in the triangle are displaceable.

For example let us consider probes originating from the edge BC. The integrally transverse vectors are $(k, 1)$ for $k \in \mathbb{Z}$. For $k = 0$, the displaceable points are the interior of the triangle BCD where $D = (0, \frac{5}{2})$. For $k > 0$, the ray from B in direction $(k, 1)$ crosses AC at $\left(\frac{15k}{5k+3}, \frac{15}{5k+3}\right)$, so the set of points displaceable by probe in this direction from BC is the interior of the triangle with vertices B, C and $\left(\frac{15k}{10k+6}, \frac{15}{10k+6}\right)$. Since for $k \geq 1$, $\left(\frac{15k}{10k+6}, \frac{15}{10k+6}\right)$ lies in BCD, we see that probes with $k \geq 1$ displace no points that are not displaceable by the vertical probes. Probes with $k = -1$ displace points in the triangle BCG where $G = (\frac{3}{2}, \frac{3}{2})$, and similar to above we can check that other probes displace no new points. Thus we have found all the points displaceable by probe from the edge BC, illustrated in Figure 10a.

Similarly we compute that the best probes from edge AB are in directions $(1, 0)$, $(1, -1)$ and $(1, -2)$, displacing points in the interiors of triangles ABE, ABG and AFB respectively, where $E = (\frac{3}{2}, 0)$ and $F = (\frac{5}{4}, \frac{5}{2})$. (See Figure 10b.)

And for the edge AC we find that the best probes are in direction $(1, -2)$, and displace points in the interior of triangle AFC. (See Figure 10c.)

This leaves an open set that is not displaceable by probes in triangle ABC. (See Figure 10d.) Smoothing the corners at A and C, for example as in Figure 9, gives a smooth polytope with an open set of points non-displaceable by probe.

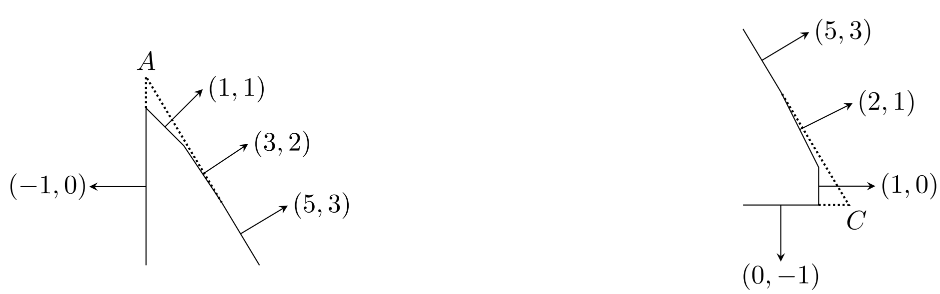
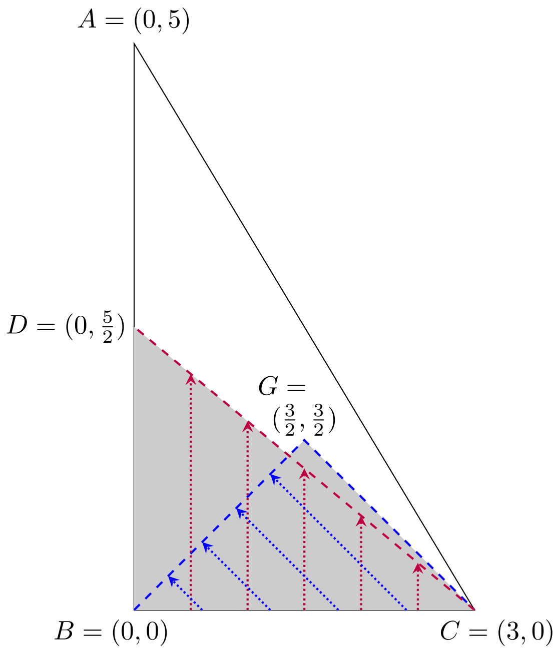
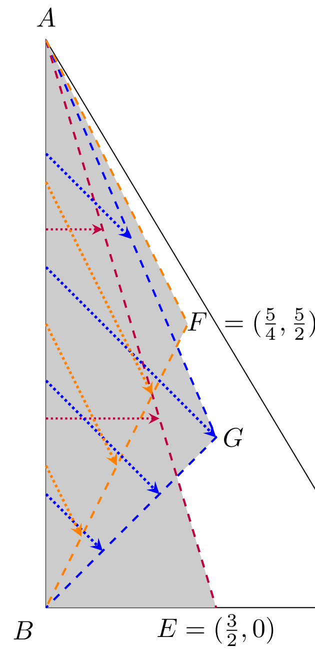


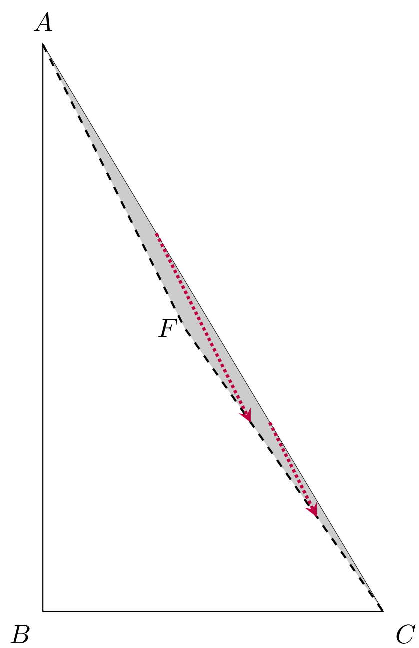
Figure 9: Smoothing the corners at A and C – two cuts are needed for each.



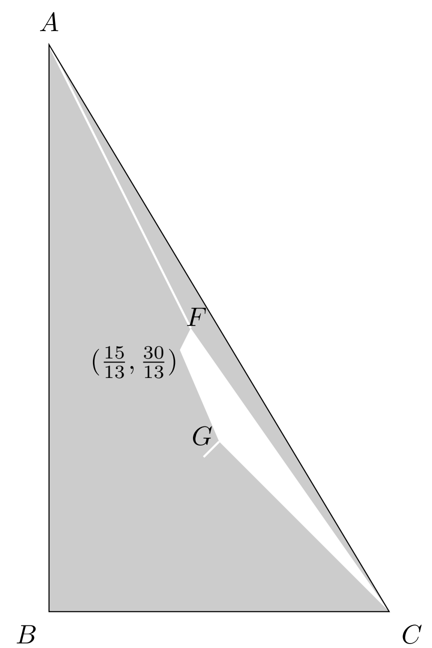
(a) Points displaceable by probes from the edge BC. Probes in direction $(0, 1)$ shown in purple, in direction $(-1, 1)$ in blue.



(b) Points displaceable by probes from the edge AB. Probes in direction $(1, 0)$ shown in purple, $(1, -1)$ in blue, $(1, -2)$ in orange.



(c) Points displaceable by probes from the edge AC. The best probes are all in direction $(-1, 2)$.



(d) The unshaded region is the set of points not displaceable by probe in triangle ABC. It contains an open set, the quadrilateral with vertices G, C, F and $(\frac{15}{13}, \frac{30}{13})$.

Figure 10: Shaded areas denote the points in triangle ABC that are displaceable by probe, showing that there is an open set that is not displaceable by probe.

4 Showing non-displaceability – symplectic reduction

In this section we discuss some of the results of [3]. We noted in Remark 2.2.4 that if M_{red} is a symplectic reduction of M, then its moment polytope $\Delta_{\mathrm{red}} = \mu_{\mathrm{red}}(M_{\mathrm{red}})$ can be seen as a planar section of the moment polytope of M, $\Delta = \mu(M)$. That means that given $x \in \mathring{\Delta}_{\mathrm{red}}$, x also lies in $\mathring{\Delta}$, and so to the toric fibre $T = \mu_{\mathrm{red}}^{-1}(x)$ corresponds the fibre $\tilde{T} = \mu^{-1}(x)$.
The method of Abreu and Macarini is based on the proposition that if $\tilde{T}$ is non-displaceable, then so is T.

In section 4.2 we show that monotone toric manifolds are symplectic reductions of a weighted projective space $\mathbb{C}\mathrm{P}(1, m_1, \ldots, m_n)$. We then use a result of Cho and Poddar [6] that says that in such weighted projective spaces the fibre at 0 is non-displaceable, to deduce that in monotone manifolds the central fibre is never displaceable.

Then in section 4.3 we consider reductions of products of manifolds and obtain a range of examples where the above method allows us to find some non-displaceable fibres.

4.1 General method

Proposition 4.1.1. *[3, Lemma 3.1]*
Suppose that (M_{red}, ω_{red}) is a a symplectic reduction of (M, ω), with notation as in Theorem 2.2.3, so $M_{red} = Z/K$ for $Z \subset M$, K a torus. Let $\pi : Z \to M_{red}$ be the projection map.

Suppose ψ^t is a Hamiltonian flow on M_{red}. Then there exists a Hamiltonian flow $\tilde{\psi}^t$ on M, such that $\tilde{\psi}^t(Z) \subset Z$ and for any $p \in Z$, $\pi(\tilde{\psi}^t(p)) = \psi^t(\pi(p))$.

Proof. Let $H_t : M_{\mathrm{red}} \to \mathbb{R}$ be the time-dependent Hamiltonian generating ψ. Then we can extend $H_t \circ \pi : Z \to \mathbb{R}$ to a smooth map $\tilde{H}_t : M \to \mathbb{R}$.

Let X_t be the vector field generating ψ_t, i.e. the vector field satisfying $\iota_{X_t}\omega_{\mathrm{red}} = \mathrm{d}H_t$, and similarly let $\tilde{X}_t$ be the vector field satisfying $\iota_{\tilde{X}_t}\omega = \mathrm{d}\tilde{H}_t$.

To show that $\tilde{\psi}^t(Z) \subset Z$ it suffices to show that for any $p \in Z$, $(\tilde{X}_t)_p \in \mathrm{T}_pZ$, and for $\pi \circ \tilde{\psi} = \psi \circ \pi$ it is enough to see that $\pi_*\tilde{X}_t = X_t$.

Now for $v \in \mathrm{T}_pZ$, if $\tilde{Y}_t$ is a lift of X_t to Z, we have

$$\omega(\tilde{X}_t, v) = \mathrm{d}\tilde{H}_t(v) \overset{\text{def of } \tilde{H}_t}{=} \mathrm{d}H_t(\pi_* v) = \omega_{\mathrm{red}}(X_t, \pi_* v) \overset{\text{def of } \omega_{\mathrm{red}}}{=} \omega(\tilde{Y}_t, v).$$

Thus $\tilde{Y}_t - \tilde{X}_t \in (\mathrm{T}_pZ)^\omega = ((\mathrm{T}_p\mathcal{O}_p)^\omega)^\omega = \mathrm{T}_p\mathcal{O}_p \subset \mathrm{T}_pZ$. Hence $\tilde{X}_t \in \mathrm{T}_pZ$ and $\pi_*\tilde{X}_t = \pi_*\tilde{Y}_t = X_t$ so we are done. □

Theorem 4.1.2. (The symplectic reduction method)*[3, Proposition 3.2]*
Suppose that (M_{red}, ω_{red}) is a a symplectic reduction of (M, ω), suppose Δ is a moment polytope of M and $\Delta_{red} \subset \Delta$ is the moment polytope of M_{red}. Suppose $x \in \mathring{\Delta}_{red}$. Let T_x be the corresponding toric fibre of M_{red} and $\tilde{T}_x$ the corresponding toric fibre of M.

If ψ^t is a Hamiltonian flow on M_{red}, and $\psi(T_x) \cap T_x = \emptyset$, then there is a Hamiltonian flow $\tilde{\psi}^t$ on M with $\tilde{\psi}(\tilde{T}_x) \cap \tilde{T}_x = \emptyset$.
Thus if T_x is displaceable, then $\tilde{T}_x$ is also displaceable.

Proof. Take the $\tilde{\psi}^t$ constructed in Proposition 4.1.1.
Suppose $q \in \tilde{\psi}(\tilde{T}_x) \cap \tilde{T}_x$. Let $p = \tilde{\psi}^{-1}(q) \in \tilde{T}_x$.
Then $\pi(p) \in T_x$ and $\psi(\pi(p)) = \pi(\tilde{\psi}(p)) = \pi(q) \in T_x$, hence $\psi(T_x) \cap T_x \neq \emptyset$. □

Remark 4.1.3. [3, Remark 3.3] We will mostly use the contrapositive of Theorem 4.1.2 to deduce non-displaceability for toric fibres on symplectic reductions. However the direct statement can also be used, and in particular McDuff's method of probes is a special case of this result. In fact the proof of the method of probes is a special case of the proof of the symplectic reduction method.

Indeed, in the coordinates of Theorem 3.1.2, we can add a constant to the moment map $\mu : M \to (\mathbb{R}^n)^*$ such that the probe p is $p = \{(x, 0, 0, \dots, 0) : 0 < x < a\}$. Then we can set $K = \mathbb{T}^{n-1}$ the last $n-1$ coordinates of $\mathbb{T}^n$, so the symplectic reduction of M by K is $\mu^{-1}(p)/\mathbb{T}^{n-1} = D^2(a)$ with the standard area form. If $x = (b, 0, \dots, 0) \in p$, then in $D^2(a)$ the fibre T_x is $\partial D^2(b)$, which is displaceable for $b < \frac{a}{2}$. This implies that the fibre $\tilde{T}_x$ in M is displaceable for $b < \frac{a}{2}$. ◊

4.2 Monotone polytopes

We will use the following fact proved in [6]:

Fact 4.2.1. *Consider the weighted projective space* $\mathbb{C}P(1, \mathbf{m}) := \mathbb{C}P(1, m_1, \dots, m_n), m_i \in \mathbf{N}$, *with the moment polytope* $\Delta_{\mathbf{m}} \subset (\mathbb{R}^n)^*$,

$$\Delta_{\mathbf{m}} = \{(x_1, \dots, x_n) : x_i \geq -1, \sum_{i=1}^{n} m_i x_i \leq 1\}.$$

Let $\mu_{\mathbf{m}} : \mathbb{C}P_{\mathbf{m}} \to \Delta_{\mathbf{m}}$ *be the corresponding moment map. Then the central fibre* $T_0 = \mu_{\mathbf{m}}^{-1}(0)$ *is non-displaceable.*

Proposition 4.2.2. *[3, Proposition 3.7]*
Suppose M *is a toric manifold,* $\nu_1, \dots, \nu_d \in \mathbb{Z}^n$ *the primitive integral interior normal vectors to the facets of the moment polytope.*
Suppose there exist positive integers $m_1, \dots, m_d$ *such that* $\sum_{i=1}^{d} m_i \nu_i = 0$, *and* $(m_1, \dots, m_d)$ *is a primitive integral vector. Then* M *is a symplectic reduction of the weighted projective space* $\mathbb{C}P(m_1, \dots, m_d)$.

Proof. By the Delzant construction, M is a reduction of $\mathbb{C}^d$ with respect to some subtorus $K \subset \mathbb{T}^d$ of dimension $d-n$, and such that the Lie algebra of K is $\mathrm{Lie}(K) = \ker \pi : \mathbb{R}^d \underset{e_i \mapsto \nu_i}{\longrightarrow} \mathbb{R}^n$.

Hence $(m_1, \dots, m_d) \in \mathrm{Lie}(K) \iff \sum_{i=1}^{d} m_i \nu_i = 0$.

Now if K' is the subtorus generated by the Lie algebra $\mathrm{Lie}(K') = \mathrm{span}\{(m_1, \dots, m_d)\} \subset \mathrm{Lie}(K)$, then the reduction of $\mathbb{C}^d$ by K' is $\mathbb{C}P(m_1, \dots, m_d)$, and so by reduction in stages, M is the reduction of $\mathbb{C}P(m_1, \dots, m_d)$ by K/K'. □

Lemma 4.2.3. *[3, Lemma 4.8]*
If M *is a monotone toric manifold with moment polytope* Δ, $\nu_1, \dots, \nu_d \in \mathbb{Z}^n$ *the primitive integral interior normals, then there exists a* $k \in \{1, \dots, d\}$ *and positive integers* m_i *such that*

$$\nu_k + \sum_{\substack{i=1,\dots,d \\ i \neq k}} m_i \nu_i = 0.$$

Proof. We use the following fact. In the algebraic language this is the fact that a toric variety is proper if and only if its fan is complete (see e.g. [7]).

Claim. *For any vector* $u \in \mathbb{R}^n$, *there is a vertex* v *of* Δ, *with facets at* v $F_1, \dots, F_n$ *and normals* $\nu_1, \dots, \nu_n$, *such that* u *lies in the cone* $Cone(\nu_1, \dots, \nu_n)$.

Proof. Since Δ is compact, the function $\Delta \to \mathbb{R},\ x \mapsto \langle x, u\rangle$ has a minimum on Δ, and by convexity this minimum will be attained at some vertex v. Let the primitive normal and edge vectors at v be $\nu_1, \dots, \nu_n$ and $\epsilon_1, \dots, \epsilon_n$. We know from the proof of Proposition 2.4.2 that $\langle \nu_i, \epsilon_j\rangle = \delta_{ij}$.

Now, for any $y \in \Delta$, $\langle y, u\rangle \geq \langle v, u\rangle$, so in particular for $i = 1, \dots, n$, $\langle \epsilon_i, u\rangle \geq 0$.

Since $\{\nu_1, \dots, \nu_n\}$ is a basis, we can write $u = \sum_{i=1}^n c_i\nu_i$ for some $c_i \in \mathbb{R}$. But as $c_i = \langle u, \epsilon_i\rangle$, all the c_i are non-negative and hence $u \in \text{Cone}(\nu_1, \dots, \nu_n)$. ☐

We now apply the above result to the vector $u = -\sum_{i=1}^d \nu_i$. Up to reordering the normal vectors, we can write $u = \sum_{i=1}^n c_i\nu_i$ with $c_i \geq 0$, and since u is an integral vector, and $\{\nu_1, \dots, \nu_n\}$ an integral basis, all the c_i are non-negative integers.

Hence, as $d > n$, we have $0 = \nu_d + \sum_{i=1}^{d-1} m_i\nu_i$ for some $m_i \in \mathbb{N}$. ☐

Putting everything together we have proved:

Theorem 4.2.4. *[3, Proposition 4.9]*
For any monotone symplectic toric manifold M, the central fibre is non-displaceable.

Proof. We have shown (Proposition 4.2.2 and Lemma 4.2.3) that M is a symplectic reduction of some weighted projective space $\mathbb{C}P(1, m_1, \dots, m_{d-1})$. Fact 4.2.1 says that in this weighted projective space the central fibre is non-displaceable, and so by the symplectic reduction method, the central fibre of M is also non-displaceable. ☐

4.3 Products

We will take advantage of the following fact (proved by Chris Woodward in [17]) to generate a range of interesting examples of applications of the symplectic reduction method.

Fact 4.3.1. *If T_i is a non-displaceable Lagrangian toric fibre of (M_i, ω_i) for $i = 1, 2$, then the fibre $T_1 \times T_2$ is non-displaceable in $(M_1 \times M_2, \omega_1 \oplus \omega_2)$.*

We also use the trivial fact that if M_i has moment polytope Δ_i, then $M_1 \times M_2$ with the product action has moment polytope $\Delta_1 \times \Delta_2$.

As ingredients to feed into this method we will use $\mathbb{C}P^n$ and also the total space $E_{k,m}$ of the line bundle $\mathcal{O}(-k) \to \mathbb{C}P^m$ with $1 \leq k \leq m$, for which we will use the following result, a special case of a general result about negative line bundles proved by Alexander Ritter and Ivan Smith in [16]:

Fact 4.3.2. *[16, Lemma 12.8, Section 12.5 & Theorem 12.16]*
If $E_{k,m}$ is the total space of the line bundle $\mathcal{O}(-k) \to \mathbb{C}P^m$ with $1 \leq k \leq m$, then $E_{k,m}$ has a moment polytope

$$\{\underline{x} \in \mathbb{R}^{m+1} : x_i \geq 0, \sum_{i=1}^m x_i - kx_{m+1} \leq 1\},$$

and the fibre at $\left(\frac{1}{1+m-k}, \frac{1}{1+m-k}, \dots, \frac{1}{1+m-k}\right)$ is non-displaceable.

Example 4.3.3. For $k = m = 1$, the moment polytope is $\{x_1 \geq 0, x_2 \geq 0, x_1 - x_2 \leq 1\}$, and the non-displaceable fibre is at (1,1). Applying the affine transformation $\underline{x} \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \underline{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, we get the moment polytope $\{x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \geq 1\}$ with non-displaceable fibre $(1, 1)$. We will use both of these forms in our examples.

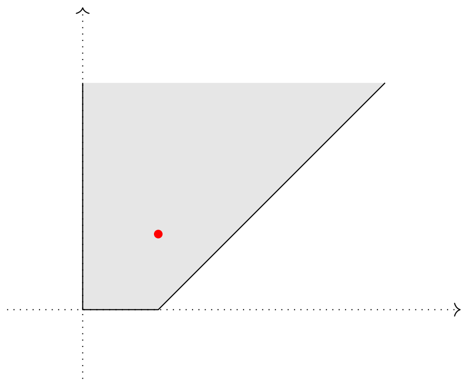
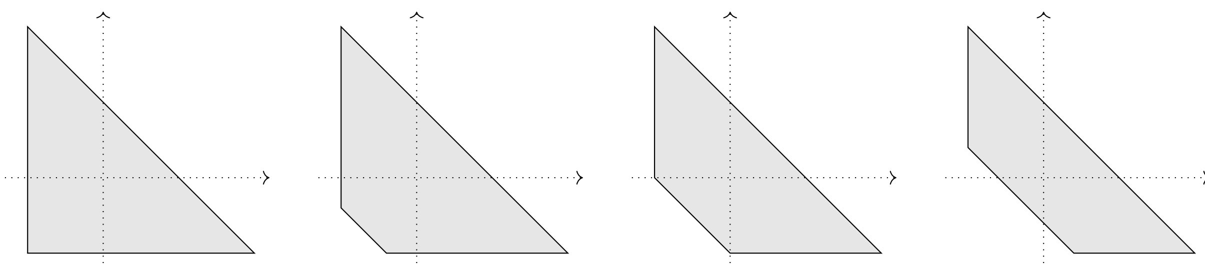


Figure 11: Two forms of the moment polytope of $E_{1,1}$ with the non-displaceable fibre marked in red.

$\Diamond$

4.3.1 Example 1: $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ [3, Application 5]

The blow-up of a toric fixed point of a toric manifold is reflected in the moment polytope as a smooth cut of the respective vertex. The length of the cut corresponds to the symplectic form on the exceptional divisor. Figure 12 shows some blow-ups of $\mathbb{C}P^2$ at one of the toric fixed points.



(a) $\mathbb{C}P^2$ (b) Small blow-up (c) Monotone blow-up (d) Large blow-up

Figure 12: Blow-ups of $\mathbb{C}P^2$.

We can view a small blow-up of $\mathbb{C}P^2$ as a symplectic reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1$:

Consider $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ with a blow-up of size $\alpha \in (0, 1)$, which has a moment polytope

$$\Delta = \{\underline{x} \in \mathbb{R}^2 : -1 \leq x_1, x_2 \leq 1,\ \alpha - 2 \leq x_1 + x_2 \leq 1\}.$$

Now consider the moment polytope of $\mathbb{C}P^2 \times \mathbb{C}P^1$, $\{x_1, x_2 \geq -1,\ x_1 + x_2 \leq 1,\ \alpha - 2 \leq x_3 \leq 2 - \alpha\}$. Then take the reduction at the level $x_1 + x_2 = x_3$ – to make sure this is valid we need to check that this plane doesn't pass through any vertices of the moment polytope (see Proposition 2.4.5). Then we can project down onto the x_1, x_2 plane, which is an integral affine transformation since the plane $x_1 + x_2 = x_3$ is integrally transversal to $(0, 0, 1)$.

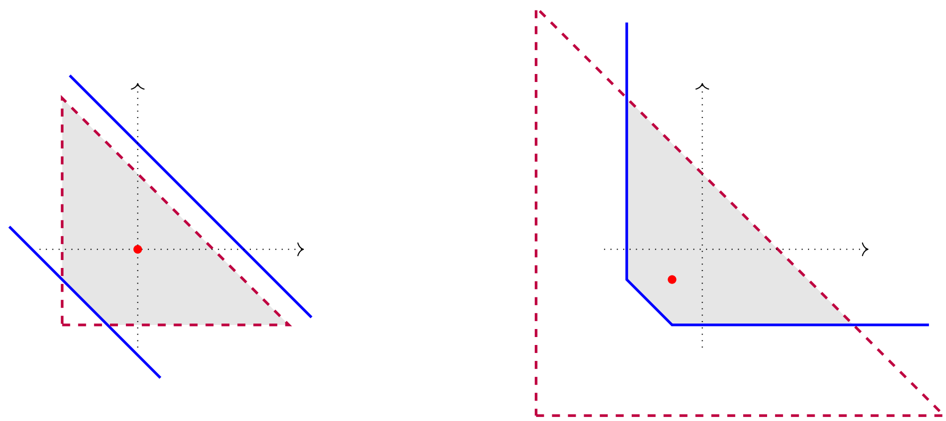
Thus we obtain Δ, and so we have expressed a small blow-up $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$ as a reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1$. As $\mathbb{C}P^2 \times \mathbb{C}P^1$ has a non-displaceable fibre at the origin, we obtain that the fibre at the origin is also non-displaceable for $\mathbb{C}P^2 \# \overline{\mathbb{C}P}^2$.

We can represent this pictorially (see Figure 13a), by projecting the moment polytopes of each factor onto the x_1, x_2 plane and showing Δ as the intersection of these projections. [3] If the non-displaceable sets of the factors intersect in some point, then this point will be a non-displaceable fibre of the reduction – in this case we obtain the fibre at the origin.

Alternatively we can view a small blow-up as a reduction of $E_{1,1} \times \mathbb{C}P^2$, by taking the following moment polytopes:

- For the $E_{1,1}$ factor take $\{x_1, x_2 \geq -1,\ x_1+x_2 \geq \alpha-2\}$. This has a non-displaceable fibre at $(\alpha-1, \alpha-1)$.
- For the $\mathbb{C}P^2$ factor take $\{x_3, x_4 \geq 3\alpha - 4,\ x_3 + x_4 \leq 1\}$. Then this has a non-displaceable fibre at $(x_3, x_4) = (\alpha - 1, \alpha - 1)$.

So taking the symplectic reduction at level $x_3 = x_1, x_4 = x_2$ gives Δ and shows that there is a non-displaceable fibre at $(\alpha - 1, \alpha - 1)$.



(a) Reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1$
($\mathbb{C}P^1$ in solid blue, $\mathbb{C}P^2$ in dashed purple).

(b) Reduction of $E_{1,1} \times \mathbb{C}P^2$
($E_{1,1}$ in solid blue, $\mathbb{C}P^2$ in dashed purple).

Figure 13: Two views of a small blow-up of $\mathbb{C}P^2$, shown for $\alpha = 0.6$. There are two non-displaceable fibres (shown as red dots), at $(0, 0)$ and $(\alpha - 1, \alpha - 1)$.

4.3.2 Example 2: $(\mathbb{C}P^1 \times \mathbb{C}P^1)\#\overline{\mathbb{C}P}^2$ – collide-and-scatter [3, Application 6]

Consider $(\mathbb{C}P^1 \times \mathbb{C}P^1)\#\overline{\mathbb{C}P}^2$, with both $\mathbb{C}P^1$ factors of equal size 1, and the blow-up of size $\alpha \in (0, 2)$. This has a moment polytope

$$\Delta = \{\underline{x} \in \mathbb{R}^2 : -1 \leq x_1, x_2 \leq 1,\ x_1 + x_2 \leq 2 - \alpha\}.$$

$\alpha = 1$ gives the monotone blow-up, whereas we refer to blow-ups with $\alpha < 1$ and $\alpha > 1$ as small and large respectively.

A small blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$ can be seen either as a reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times CP^1$, or as a reduction of $E_{1,1} \times \mathbb{C}P^2$, thus giving us two non-displaceable fibres, at $(0, 0)$ and $(1-\alpha, 1-\alpha)$. See Figure 14 for illustration.

[3] To check that the plane of the cut does not intersect any faces of codimension greater than 2 (the condition we need from Proposition 2.4.5), it is enough to check that in our picture no edge of one factor passes through a vertex of another factor.

Small blow-up as reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$:
Take the following moment polytopes:

- For the $\mathbb{C}P^2$ factor take $\{x_1, x_2 \geq \alpha - 2,\ x_1 + x_2 \leq 2 - \alpha\}$. This has a non-displaceable fibre at $(0, 0)$.
- For the $\mathbb{C}P^1$ factors, take $\{-1 \leq x_i \leq 1\}$ for $i = 3, 4$. Then these have non-displaceable fibres at $x_i = 0$.

Hence taking the reduction at level $x_3 = x_1,\ x_4 = x_2$ we obtain Δ with a non-displaceable fibre at $(0, 0)$.

Small blow-up as reduction of $E_{1,1} \times \mathbb{C}P^2$:
Take the following moment polytopes:

- For the $E_{1,1}$ factor, take $\{\leq x_1, x_2 \leq 1,\ x_1 + x_2 \leq 2 - \alpha\}$.This has a non-displaceable fibre at $(1-\alpha, 1-\alpha)$.
- For the $\mathbb{C}P^2$ factor take $\{x_3, x_4 \geq -1,\ x_3 + x_4 \leq 4 - 3\alpha\}$. This has a non-displaceable fibre at $(x_3, x_4) = (1 - \alpha, 1 - \alpha)$.

Hence taking the reduction at level $x_3 = x_1,\ x_4 = x_2$ we obtain Δ with a non-displaceable fibre at $(1-\alpha, 1-\alpha)$.

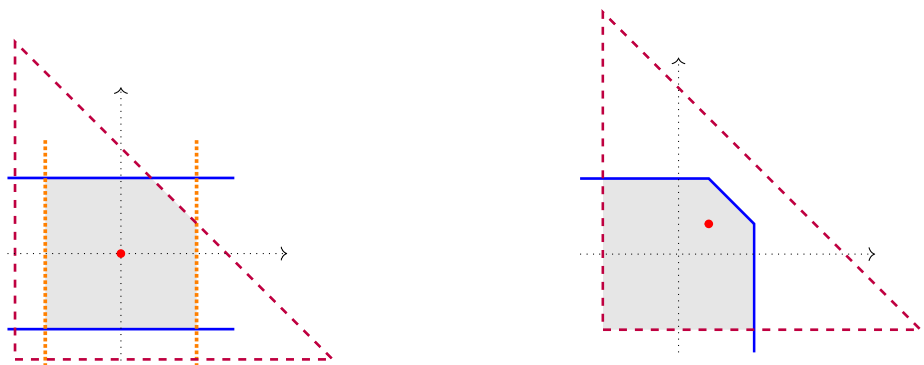


Figure 14: Small blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$ (shown for $\alpha = 0.6$) as reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ (left) and $E_{1,1} \times \mathbb{C}P^2$ (right). (Left: $\mathbb{C}P^2$ in dashed purple, $\mathbb{C}P^1$s in solid blue and dotted orange. Right: $\mathbb{C}P^2$ in dashed purple, $E_{1,1}$ in solid blue.)

For a large blow-up however, we get three non-displaceable fibres, by considering it as reductions of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ and of $E_{1,1} \times \mathbb{C}P^1 \times \mathbb{C}P^1$ in two ways. (See Figure 15.)

Large blow-up as reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$:
Take the following moment polytopes:

- For the $\mathbb{C}P^2$ factor take $\{x_1, x_2 \geq -1,\ x_1 + x_2 \leq 2 - \alpha\}$. This has a non-displaceable fibre at $(\frac{1-\alpha}{3}, \frac{1-\alpha}{3})$.
- For the $\mathbb{C}P^1$ factors, take $\{\frac{-1-2\alpha}{3} \leq x_i \leq 1\}$ for $i = 3, 4$. Then these have non-displaceable fibres at $x_i = \frac{1-\alpha}{3}$.

Hence taking the reduction at level $x_3 = x_1,\ x_4 = x_2$ we obtain Δ with a non-displaceable fibre at $(\frac{1-\alpha}{3}, \frac{1-\alpha}{3})$.

Large blow-up as reduction of $E_{1,1} \times \mathbb{C}P^1 \times \mathbb{C}P^1$:
This is one of the two ways, the other is obtained by interchanging x_1 and x_2.

Take the following moment polytopes:

- For the $E_{1,1}$ factor, take $\{x_1 \geq -1,\ x_2 \leq 1,\ x_1 + x_2 \leq 2 - \alpha\}$.This has a non-displaceable fibre at $(1 - \alpha, \alpha - 1)$.
- For the first $\mathbb{C}P^1$ factor take $\{1 - 2\alpha \leq x_3 \leq 1\}$. This has a non-displaceable fibre at $x_3 = 1 - \alpha$.

- For the second $\mathbb{C}P^1$ factor take $\{-1 \le x_4 \le 2\alpha - 1\}$. This has a non-displaceable fibre at $x_4 = \alpha - 1$.

Hence taking the reduction at level $x_3 = x_1,\ x_4 = x_2$ we obtain Δ with a non-displaceable fibre at $(1-\alpha, \alpha-1)$.

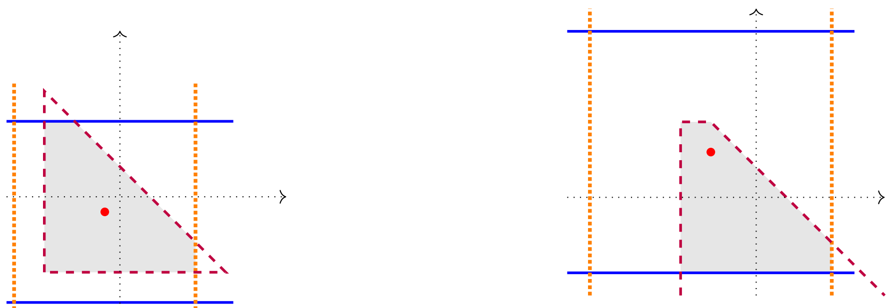


Figure 15: Large blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$ (shown for $\alpha = 1.6$) as reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ and $E_{1,1} \times \mathbb{C}P^1 \times \mathbb{C}P^1$. (Left: $\mathbb{C}P^2$ in dashed purple, $\mathbb{C}P^1$s in solid blue and dotted orange. Right: $E_{1,1}$ in dashed purple, $\mathbb{C}P^1$s in solid blue and dotted orange.)

This is an example of the collide-and-scatter phenomenon: as the size of the blow-up grows, the two non-displaceable fibres get closer together, until they collide when the blow-up is monotone and then scatter into three.

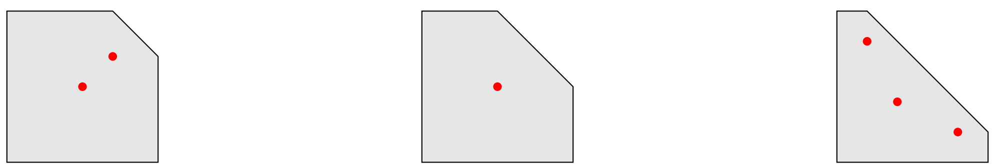


Figure 16: Collide-and-scatter phenomenon for the blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$. The figure shows a small, monotone and large blow-up, which have respectively two, one and three non-displaceable fibres.

4.3.3 Example 3: $(\mathbb{C}P^1 \times \mathbb{C}P^1)\#\overline{\mathbb{C}P}^2$ – interval of non-displaceable fibres [3, Application 7]

Consider a small blow-up of $\mathbb{C}P^1 \times \mathbb{C}P^1$, with one factor bigger than the other. Then expressing it as a reduction of $E_{1,1} \times \mathbb{C}P^1 \times \mathbb{C}P^1$, we have one degree of freedom for choosing the size of the $E_{1,1}$ factor, which gives us a closed interval of non-displaceable fibres.

Let the moment polytope be

$$\Delta = \{-2 \le x_1 \le 2,\ -1 \le x_2 \le 1,\ x_1 + x_2 \le 2\}.$$

The interval of non-displaceable fibres will then be $\{(t, 0) : t \in [0, 1]\}$. To check that $(t, 0)$ is non-displaceable, we take:

- For the $E_{1,1}$ factor take $\{x_1 \le 2, x_2 \le 2 + t, x_1 + x_2 \le 2\}$. This has a non-displaceable fibre at $(t, 0)$.
- For the first $\mathbb{C}P^1$ take $\{-2 \le x_3 \le 2 + 2t\}$. This has a non-displaceable fibre at $x_3 = t$.
- For the second $\mathbb{C}P^1$ take $\{-1 \le x_4 \le 1\}$. This has a non-displaceable fibre at $x_4 = 0$.

Then taking the symplectic reduction at the level $x_3 = x_1, x_4 = x_2$, we obtain Δ with a non-displaceable fibre at $(t, 0)$.

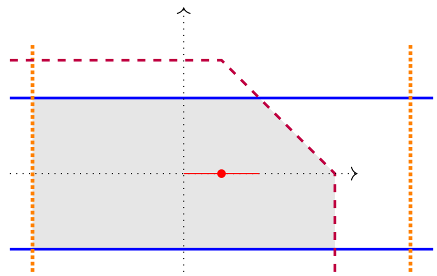


Figure 17: A closed interval of non-displaceable fibres on $(\mathbb{C}P^1 \times \mathbb{C}P^1)\#\overline{\mathbb{C}P}^2$. ($E_{1,1}$ in dashed purple, $\mathbb{C}P^1$s in solid blue and dotted orange.)

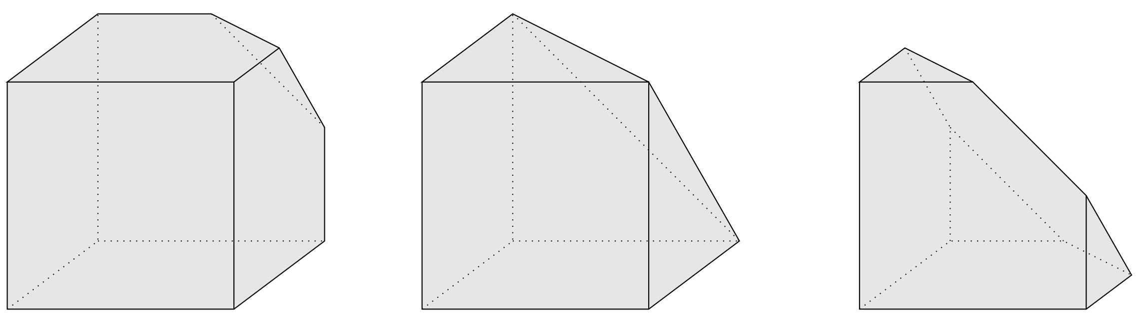
4.3.4 Example 4: Truncated cube – collide-and-scatter in three dimensions

We can perform an analogous construction to Example 2 in three dimensions, and we obtain a similar collide-and-scatter phenomenon. However while in two dimensions there was a smooth transition from a small blow-up via the monotone polytope to the large blow-up, here the polytope that would be monotone is not smooth.

Consider the polytopes

$$\Delta = \{0 \le x_1, x_2, x_3 \le 1, x_1 + x_2 + x_3 \le 3 - a\} \quad \text{for} \quad 0 < a < 2,$$

which are sections of the cube $[0,1] \times [0,1] \times [0,1]$ by planes with normal vector $(1,1,1)$. For $0 < a < 1$ these correspond to a blow-up of $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1 \times \mathbb{C}P^1$ at one point, with the blow-up of size a, whereas for $1 < a < 2$ these are blow-ups of $\mathbb{C}P^3$ at three points, of equal size $2 - a$. Finally $a = 1$ gives a non-smooth polytope with (after translation) all structure constants equal.



(a) $a < 1$. (b) $a = 1$. (c) $a > 1$.

Figure 18: Sections of the cube for various values of a. Note that the section with $a = 1$ is not smooth.

Small sections: $a < 1$

For small a we can consider the blow-up $(\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1)\#\overline{\mathbb{C}P}^3$ either as a reduction of $\mathbb{C}P^3 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ or as a reduction of $\mathbb{C}P^3 \times E_{1,2}$, thus giving us two non-displaceable fibres, at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $\left(1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2}\right)$ respectively. This is analogous to the 2-dimensional case shown in Figure 14.

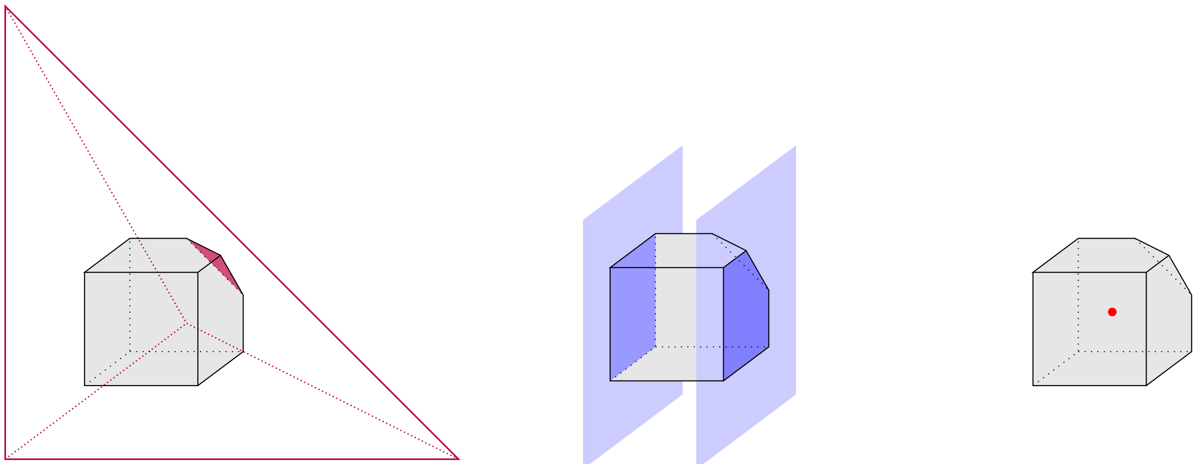
As reduction of $\mathbb{C}P^3 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$

Take the following moment polytopes:

- For $\mathbb{C}P^3$ take $\{x_1, x_2, x_3 \ge a - 1,\ x_1 + x_2 + x_3 \le 3 - a\}$. Then this has a non-displaceable fibre at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

- For the $\mathbb{C}P^1$ factors, take $\{0 \leq x_i \leq 1\}$ for $i = 4, 5, 6$. Then these all have a non-displaceable fibre at $x_i = \frac{1}{2}$.

Then taking the slice $x_1 = x_4,\ x_2 = x_5,\ x_3 = x_6$, we obtain Δ, with a non-displaceable fibre at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.



(a) The $\mathbb{C}P^3$ factor forms the triangular face of Δ.

(b) One of the $\mathbb{C}P^1$ factors. Each $\mathbb{C}P^1$ factor forms a pair of opposite faces of Δ.

(c) The non-displaceable fibre.

Figure 19: $(\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1)\#\overline{\mathbb{C}P}^3$ as reduction of $\mathbb{C}P^3 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, with blow-up of size $a = \frac{1}{2}$. Shown in the diagram are the moment polytopes of the factors and the non-displaceable fibre. We only show one of the $\mathbb{C}P^1$ factors, the other two are similar. The shaded facets of the truncated cube in (a) and (b) are those facets which are formed by the respective factor.

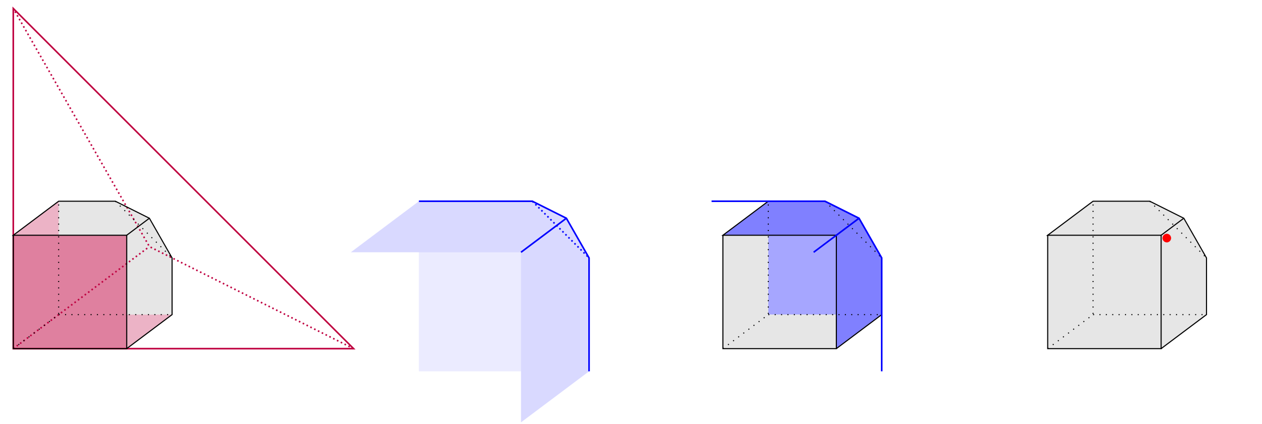
As reduction of $\mathbb{C}P^3 \times E_{1,2}$

A computation similar to Example 4.3.3 shows that one form of the moment polytope of $E_{1,2}$ is $\{x_1, x_2, x_3 \geq 0,\ x_1 + x_2 + x_3 \geq 1\}$, with non-displaceable fibre at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$.

Take the following moment polytopes:

- For $\mathbb{C}P^3$ take $\{x_1, x_2, x_3 \geq 0,\ x_1 + x_2 + x_3 \leq 4 - 2a\}$. Then this has a non-displaceable fibre at $\left(1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2}\right)$.
- For the $E_{1,2}$, take $\{x_4, x_5, x_6 \leq 1,\ x_4 + x_5 + x_6 \leq 3 - a\}$. Then this has a non-displaceable fibre at $(x_4, x_5, x_6) = \left(1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2}\right)$.

Then taking the slice $x_1 = x_4,\ x_2 = x_5,\ x_3 = x_6$ again, we obtain Δ, with a non-displaceable fibre at $\left(1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2}\right)$.



(a) The $\mathbb{C}P^3$ factor forms the three square faces of Δ.

(b) The $E_{1,2}$ factor (without truncated cube).

(c) The $E_{1,2}$ factor forms the triangular face and the three pentagonal faces of Δ.

(d) The non-displaceable fibre.

Figure 20: $(\mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1)\#\overline{\mathbb{C}P}^3$ as reduction of $\mathbb{C}P^3 \times E_{1,2}$, shown here for $a = \frac{1}{2}$.

Large sections: $a > 1$

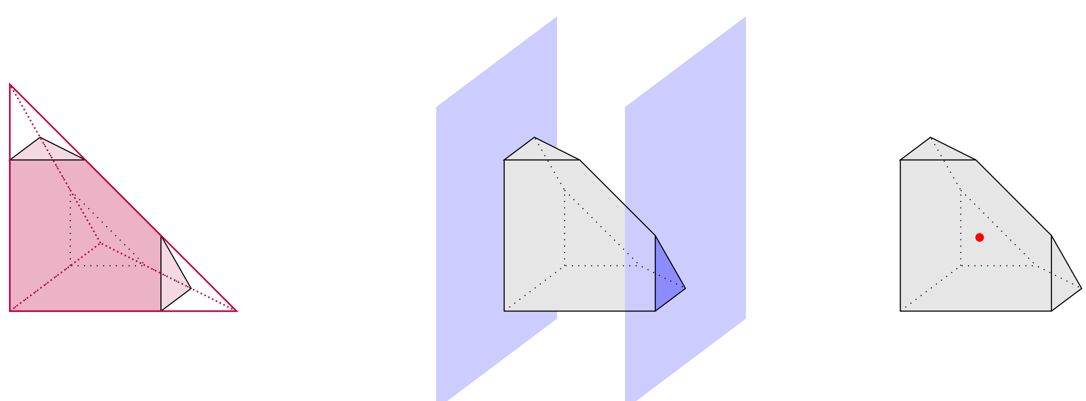
For large a, the truncated cube is the moment polytope of $\mathbb{C}P^3\#3\overline{\mathbb{C}P}^3$ with the three blow-ups of equal size. We can consider this either as a reduction of $\mathbb{C}P^3 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, or as a reduction of $E_{1,2} \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$ (in three different ways), thus giving us four non-displaceable fibres: $\left(\frac{3-a}{4}, \frac{3-a}{4}, \frac{3-a}{4}\right)$ from the first view and $\left(\frac{a}{2}, 1-\frac{a}{2}, 1-\frac{a}{2}\right)$, $\left(1-\frac{a}{2}, \frac{a}{2}, 1-\frac{a}{2}\right)$, $\left(1-\frac{a}{2}, 1-\frac{a}{2}, \frac{a}{2}\right)$ from the second view. This is analogous to the 2-dimensional case shown in Figure 15.

As reduction of $\mathbb{C}P^3 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$

Take the following moment polytopes:

- For $\mathbb{C}P^3$ take $\{x_1, x_2, x_3 \geq 0,\ x_1 + x_2 + x_3 \leq 3 - a\}$. Then this has a non-displaceable fibre at $\left(\frac{3-a}{4}, \frac{3-a}{4}, \frac{3-a}{4}\right)$.
- For the $\mathbb{C}P^1$, take $\{\frac{1-a}{2} \leq x_i \leq 1\}$ for $i = 4, 5, 6$. Then these all have a non-displaceable fibre at $x_i = \frac{3-a}{4}$.

Taking the slice $x_1 = x_4,\ x_2 = x_5,\ x_3 = x_6$, we obtain Δ, with a non-displaceable fibre at $\left(\frac{3-a}{4}, \frac{3-a}{4}, \frac{3-a}{4}\right)$.



(a) The $\mathbb{C}P^3$ factor forms the pentagonal and hexagonal faces of the truncated cube.

(b) The $\mathbb{C}P^1$ factors form the triangular faces of the truncated cube.

(c) The non-displaceable fibre.

Figure 21: $\mathbb{C}P^3\#3\overline{\mathbb{C}P}^3$ as reduction of $\mathbb{C}P^3 \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, shown for $a = \frac{3}{2}$.

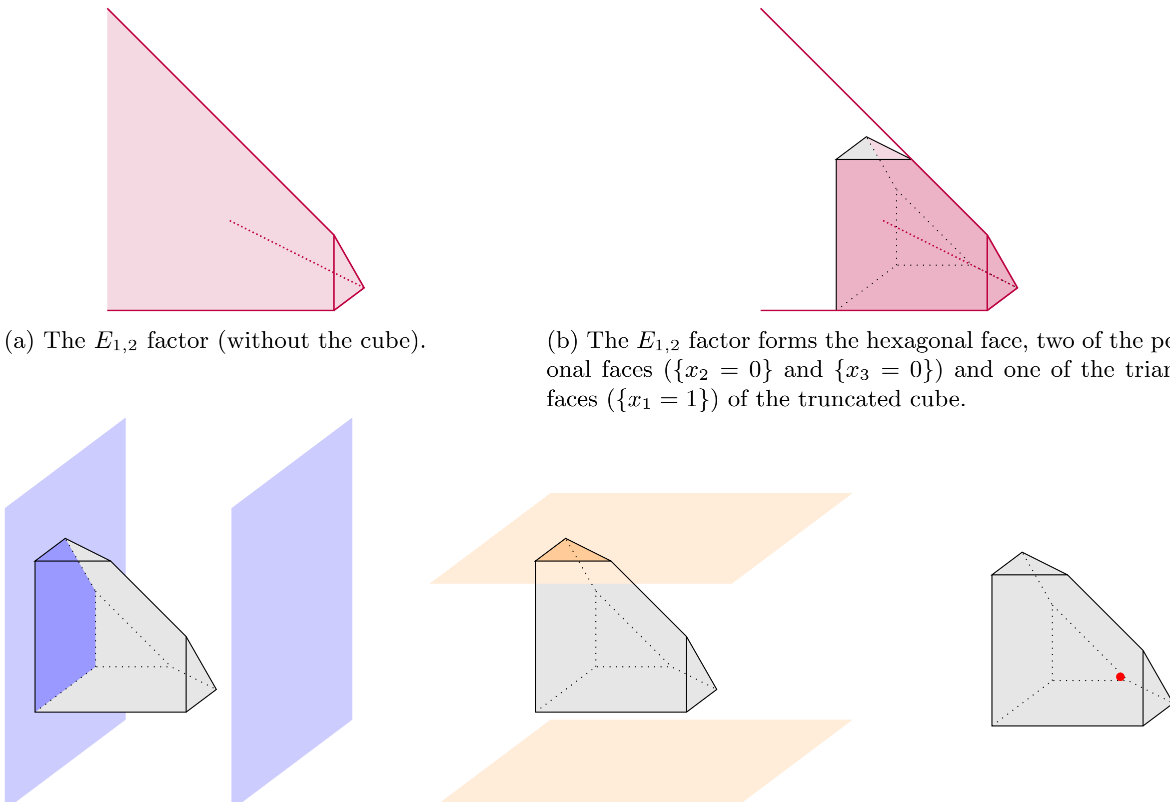
As reduction of $E_{1,2} \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$

There are three ways of expressing $\mathbb{C}P^3 \# 3\overline{\mathbb{C}P}^3$ as a reduction of $E_{1,2} \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, with distinguished coordinate x_1, x_2, x_3 respectively. Here we show the case with distinguished coordinate x_1.

Take the following moment polytopes:

- For $E_{1,2}$ take $\{x_1 \leq 1,\ x_2 \geq 0,\ x_3 \geq 0,\ x_1 + x_2 + x_3 \leq 3 - a\}$. Then this has a non-displaceable fibre at $\left(\frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2}\right)$.
- For the first $\mathbb{C}P^1$, take $\{0 \leq x_4 \leq a\}$, which has a non-displaceable fibre at $x_4 = \frac{a}{2}$.
- For the other two $\mathbb{C}P^1$ factors, take $\{1 - a \leq x_i \leq 1\}$, for $i = 5, 6$. Then these have non-displaceable fibres at $x_i = 1 - \frac{a}{2}$.

Taking the slice $x_1 = x_4,\ x_2 = x_5,\ x_3 = x_6$, we obtain Δ, with a non-displaceable fibre at $\left(\frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2}\right)$.



(a) The $E_{1,2}$ factor (without the cube).

(b) The $E_{1,2}$ factor forms the hexagonal face, two of the pentagonal faces ($\{x_2 = 0\}$ and $\{x_3 = 0\}$) and one of the triangular faces ($\{x_1 = 1\}$) of the truncated cube.

(c) The first $\mathbb{C}P^1$ factor forms the remaining pentagonal face ($\{x_1 = 0\}$).

(d) The other $\mathbb{C}P^1$ factors forms the other two triangular faces.

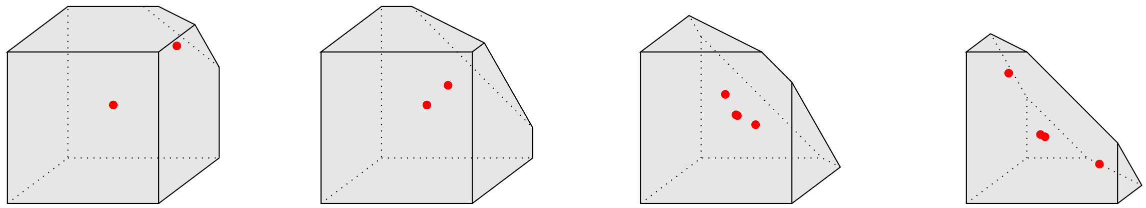
(e) The non-displaceable fibre.

Figure 22: $\mathbb{C}P^3 \# 3\overline{\mathbb{C}P}^3$ as reduction of $E_{1,2} \times \mathbb{C}P^1 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, shown for $a = \frac{3}{2}$.

Thus the collide-and-scatter pattern looks as follows:

- For $a < 1$, there are two non-displaceable fibres, at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $\left(1 - \frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2}\right)$.
- For $a > 1$, there are four non-displaceable fibres, at $\left(\frac{3-a}{4}, \frac{3-a}{4}, \frac{3-a}{4}\right)$, $\left(\frac{a}{2}, 1 - \frac{a}{2}, 1 - \frac{a}{2}\right)$, $\left(1 - \frac{a}{2}, \frac{a}{2}, 1 - \frac{a}{2}\right)$ and $\left(1 - \frac{a}{2}, 1 - \frac{a}{2}, \frac{a}{2}\right)$.

Note that all of these agree when substituting $a = 1$, so this is similar to the two-dimensional collide-and-scatter pattern except for the polytope with $a = 1$ not being smooth.



(a) $a = 0.4$. (b) $a = 0.8$. (c) $a = 1.2$. (d) $a = 1.6$.

Figure 23: Collide-and-scatter pattern in 3 dimensions. For $a < 1$ there are two non-displaceable fibres, whereas for $a > 1$ there are four. (There are four non-displaceable fibres in (c) and (d), though some look merged together as an artefact of projecting the three-dimensional polytope onto a plane.)

4.3.5 Example 5: $(\mathbb{C}P^2 \times \mathbb{C}P^1)\#\overline{\mathbb{C}P}^3$ – segment of non-displaceable fibres

Consider the toric manifold M which is a product of $\mathbb{C}P^2$ and $\mathbb{C}P^1$, with $\mathbb{C}P^2$ of size 1 and $\mathbb{C}P^1$ of size b, blown-up at a torus fixed point, with blow-up of size $a < \min(1, b)$. A moment polytope of M is

$$\Delta = \{x_1, x_2, x_3 \geq 0,\ x_2 \leq b,\ x_1 + x_3 \leq 1,\ x_1 + x_2 + x_3 \geq a\}.$$

We will demonstrate three separate ways of writing M as a reduction of a product, valid for different values of a. We will see that there are two non-displaceable fibres for $a \leq \frac{1}{3}$, one for $\frac{1}{3} < a < \frac{2}{3}$ and a closed segment of non-displaceable fibres for $a = \frac{2}{3}$.

Label the six faces of Δ as follows:
- $F_i = \{x_i = 0\} \cap \Delta$ for $i = 1, 2, 3$.
- $F_4 = \{x_1 + x_2 + x_3 = a\} \cap \Delta$.
- $F_5 = \{x_1 + x_3 = 1\} \cap \Delta$.
- $F_6 = \{x_2 = b\} \cap \Delta$.

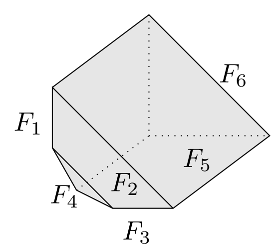


Figure 24: The moment polytope Δ of $(\mathbb{C}P^2 \times \mathbb{C}P^1)\#\overline{\mathbb{C}P}^3$, here with $\mathbb{C}P^1$ of size $b = 2$ and blow-up of size $a = \frac{1}{2}$.

View 1: as reduction of $E_{1,2} \times \mathbb{C}P^2 \times \mathbb{C}P^1$, with $a \leq \frac{2}{3}$.

Here the faces F_1, F_2, F_3, F_4 are formed by $E_{1,2}$, $\mathbb{C}P^2$ forms the face F_5 and $\mathbb{C}P^1$ forms the face F_6.

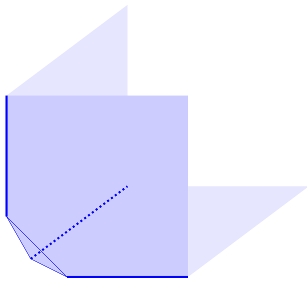
We obtain a non-displaceable fibre at $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$.

Take the following moment polytopes:

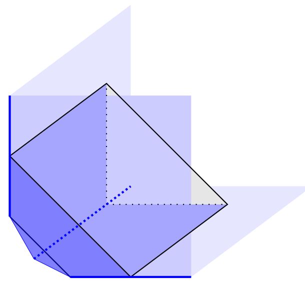
- For $E_{1,2}$ take $\{x_1, x_2, x_3 \geq 0,\ x_1 + x_2 + x_3 \geq a\}$. This has a non-displaceable fibre at $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$.
- For $\mathbb{C}P^2$ take $\{x_4, x_5 \geq \frac{3a}{4} - \frac{1}{2},\ x_4 + x_5 \leq 1\}$. This has a non-displaceable fibre at $(x_4, x_5) = \left(\frac{a}{2}, \frac{a}{2}\right)$.
- For $\mathbb{C}P^1$ take $\{a - b \leq x_6 \leq b\}$. This has a non-displaceable fibre at $x_6 = \frac{a}{2}$.

Then taking the slice $x_4 = x_1$, $x_5 = x_3$, $x_6 = x_2$ gives us Δ, with a non-displaceable fibre at $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$.

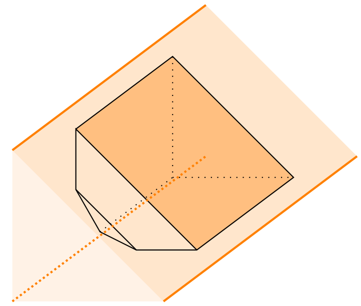
Note that we need the condition $a \leq \frac{2}{3}$ for the $\mathbb{C}P^2$ factor to contain Δ.



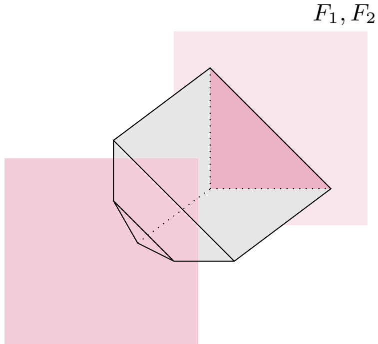
(a) $E_{1,2}$ (shown without Δ).



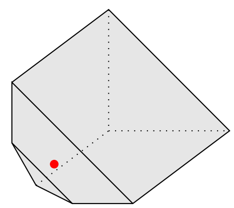
(b) The $E_{1,2}$ factor forms the faces F_1, F_2, F_3, F_4 of Δ.



(c) The $\mathbb{C}P^2$ factor forms the face F_5 of Δ.



(d) The $\mathbb{C}P^1$ factor forms the face F_6 of Δ.



(e) The non-displaceable fibre.

Figure 25: M as symplectic reduction of $E_{1,2} \times \mathbb{C}P^2 \times \mathbb{C}P^1$, shown here with $a = \frac{1}{2}, b = 2$.

View 2: as reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, with $a \leq \frac{1}{3}$.

Here the faces F_1, F_3, F_5 are formed by the $\mathbb{C}P^2$, the first $\mathbb{C}P^1$ forms the face F_4 and the second $\mathbb{C}P^1$ forms the faces F_2, F_6.

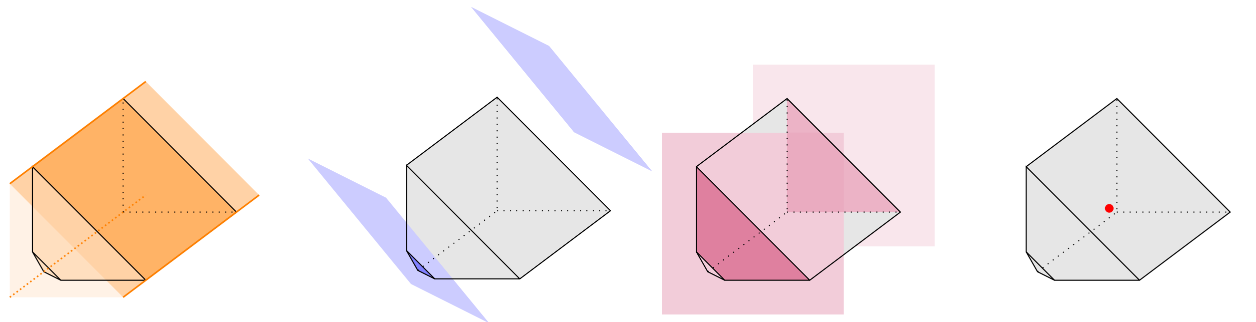
We obtain a non-displaceable fibre at $\left(\frac{1}{3}, \frac{b}{2}, \frac{1}{3}\right)$.

Take the following moment polytopes:

- For $\mathbb{C}P^2$ take $\{x_1, x_3 \geq 0,\ x_1 + x_3 \leq 1\}$. This has a non-displaceable fibre at $(x_1, x_3) = \left(\frac{1}{3}, \frac{1}{3}\right)$.
- For the first $\mathbb{C}P^1$ take $\{a \leq x_4 \leq b - a + \frac{4}{3}\}$. This has a non-displaceable fibre at $x_4 = \frac{b}{2} + \frac{2}{3}$.
- For the second $\mathbb{C}P^1$ take $\{0 \leq x_2 \leq b\}$. This has a non-displaceable fibre at $x_2 = \frac{b}{2}$.

Then taking the slice $x_4 = x_1 + x_2 + x_3$ gives us Δ, with a non-displaceable fibre at $\left(\frac{1}{3}, \frac{b}{2}, \frac{1}{3}\right)$.

Note that the edge between F_2 and F_5 lies in the plane $x_1 + x_2 + x_3 = b + 1$, hence we need the condition $a \leq \frac{1}{3}$ for the first $\mathbb{C}P^1$ to contain Δ. It is somewhat counter-intuitive but the value of b does not matter.



(a) The $\mathbb{C}P^2$ factor forms the faces F_1, F_3, F_5 of Δ.

(b) The first $\mathbb{C}P^1$ factor forms the face F_4 of Δ.

(c) The second $\mathbb{C}P^1$ factor forms the faces F_2, F_6 of Δ.

(d) The non-displaceable fibre.

Figure 26: M as symplectic reduction of $\mathbb{C}P^2 \times \mathbb{C}P^1 \times \mathbb{C}P^1$, shown here with $a = \frac{1}{4}, b = 2$.

View 3: as reduction of $E_{1,2} \times \mathbb{C}P^2 \times \mathbb{C}P^1$, with $a = \frac{2}{3}$.

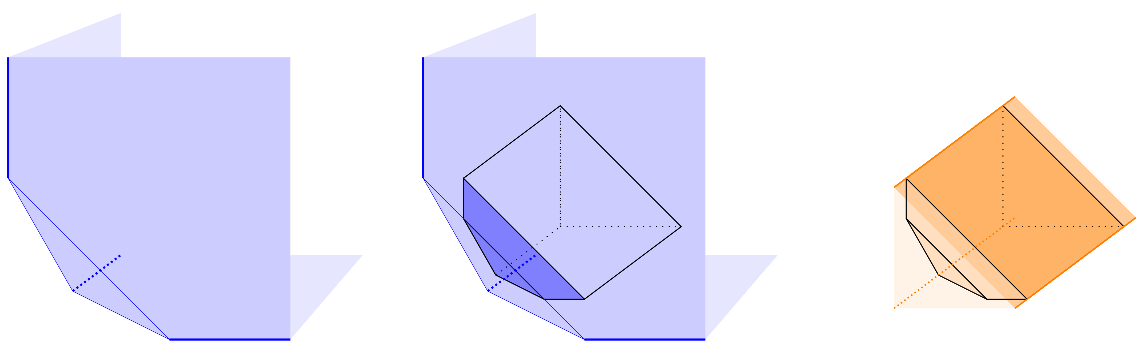
Here the faces F_2, F_4 are formed by $E_{1,2}$, $\mathbb{C}P^2$ forms the faces F_1, F_3, F_5 and $\mathbb{C}P^1$ forms the face F_6. We have one degree of freedom in choosing the size of the $E_{1,2}$ factor, and so obtain a segment of non-displaceable fibres at $\left(\frac{1}{3}, \frac{1}{3} + s, \frac{1}{3}\right)$ for $s \in \left[0, \frac{b}{2} - \frac{1}{3}\right]$.

Take the following moment polytopes, with parameter $s \in \left[0, \frac{b}{2} - \frac{1}{3}\right]$.

- For $E_{1,2}$ take $\{x_1, x_3 \geq s,\ x_2 \geq 0,\ x_1 + x_2 + x_3 \geq a\}$.
  This has a non-displaceable fibre at $\left(\frac{a}{2}, \frac{a}{2} + s, \frac{a}{2}\right) = \left(\frac{1}{3}, \frac{1}{3} + s, \frac{1}{3}\right)$.
- For $\mathbb{C}P^2$ take $\{x_4, x_5 \geq 0,\ x_4 + x_5 \leq 1\}$. This has a non-displaceable fibre at $(x_4, x_5) = \left(\frac{1}{3}, \frac{1}{3}\right)$.
- For $\mathbb{C}P^1$ take $\{\frac{2}{3} + 2s - b \leq x_6 \leq b\}$. This has a non-displaceable fibre at $x_6 = \frac{1}{3} + s$.

Then taking the slice $x_4 = x_1$, $x_5 = x_3$, $x_6 = x_2$ gives us Δ, with a non-displaceable fibre at $\left(\frac{1}{3}, \frac{1}{3} + s, \frac{1}{3}\right)$.

Note that for this to be valid we need $\frac{2}{3} + 2s - b \leq 0$, which is where the condition $s \leq \frac{b}{2} - \frac{1}{3}$ comes from.



(a) $E_{1,2}$ (shown without Δ).

(b) The $E_{1,2}$ factor forms the faces F_2, F_4 of Δ.

(c) The $\mathbb{C}P^2$ factor forms the faces F_1, F_3, F_5 of Δ.

Figure 27: One-parameter family of presentations of M with $a = \frac{2}{3}$ as a symplectic reduction of $E_{1,2} \times \mathbb{C}P^2 \times \mathbb{C}P^1$, shown here with $b = 2$ and parameter $s = \frac{1}{3}$.

(d) The $\mathbb{C}P^1$ factor forms the face F_6 of Δ.

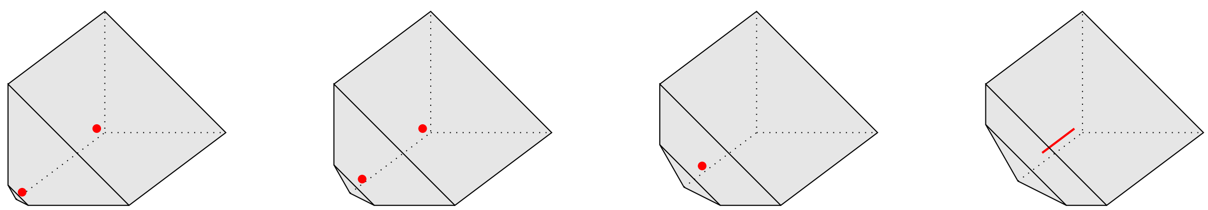
(e) The segment of non-displaceable fibres for $s \in [0, \frac{b}{2} - \frac{1}{3}]$, the fibre with $s = \frac{1}{3}$ is shown by the dot.

Figure 27: One-parameter family of presentations of M with $a = \frac{2}{3}$ as a symplectic reduction of $E_{1,2} \times \mathbb{C}P^2 \times \mathbb{C}P^1$, shown here with $b = 2$ and parameter $s = \frac{1}{3}$.

To summarise the non-displaceable fibres that we have found:

- For $a \leq \frac{2}{3}$: $\left(\frac{a}{2}, \frac{a}{2}, \frac{a}{2}\right)$.
- For $a \leq \frac{1}{3}$: $\left(\frac{1}{3}, \frac{b}{2}, \frac{1}{3}\right)$.
- For $a = \frac{2}{3}$: $\left(\frac{1}{3}, \frac{1}{3} + s, \frac{1}{3}\right)$ for any $s \in [0, \frac{b}{2} - \frac{1}{3}]$.

We show in Figure 28 the non-displaceable fibres for the values $a = \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$.



(a) $a = \frac{1}{6}$. (b) $a = \frac{1}{3}$. (c) $a = \frac{1}{2}$. (d) $a = \frac{2}{3}$.

Figure 28: The non-displaceable fibres of $M = (\mathbb{C}P^2 \times \mathbb{C}P^1)\#\overline{\mathbb{C}P}^3$, with the blow-up of varying size a.

Note that the fact that $\left(\frac{1}{3}, \frac{b}{2}, \frac{1}{3}\right)$ is non-displaceable for $a \leq \frac{1}{3}$ and also for $a = \frac{2}{3}$ suggests that it is likely to similarly be non-displaceable for $\frac{1}{3} < a < \frac{2}{3}$. (At the very least it is not displaceable by probe.) Further none of our presentations tell us anything about large blow-ups of size $a > \frac{2}{3}$. It is likely that there are more presentations of M as a reduction of a product remaining to be found, but also possible that not all non-displaceable fibres are accessible by this method.

5 Conclusion

We have discussed two combinatorial methods that can be used to determine whether or not a Lagrangian toric fibre is displaceable by Hamiltonian isotopy.

These methods are particularly useful when dealing with monotone symplectic toric manifolds. Indeed the symplectic reduction method gave us that the central fibre of a monotone toric manifold is never displaceable, and using the method of probes we saw that in the monotone case the displaceability of all fibres but the central fibre is equivalent to a purely combinatorial condition, the star Ewald condition. This is enough to prove that for all monotone symplectic toric manifolds of dimension $2n \leq 10$ the central fibre is the unique non-displaceable fibre. [4]

However for higher dimensions the question whether or not the central fibre of a monotone toric manifold is the unique non-displaceable toric fibre is still open, and so even in the monotone case the question of displaceability has not yet been fully resolved.

The question of displaceability of Lagrangian toric fibres continues to attract significant interest, and while ignoring the many recent Floer-theoretic developments in the area, we will very briefly mention two developments that are directly related to the contents of this exposition. Firstly, in [2] Abreu, Borman and McDuff introduced *extended probes*, generalising the method of probes to probes that are no longer simply straight line segments, but consist of multiple connected line segments, and are obtained by *deflecting* a probe by other probes. Using extended probes they show that in the example in Section 3.3 all fibres except those belonging to a line segment and a few more isolated ones can be displaced. This illustrates that the method of extended probes is more powerful than the plain method of probes.

Secondly, in [13], Marinković and Pabiniak prove that all toric manifolds can be realised as a centred reduction of a product of weighted projective spaces, and thus the method of Abreu and Macarini implies that all symplectic toric manifolds have at least one non-displaceable toric fibre.

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