Catalan Numbers

September 2017

I $2n$ points are positioned in a circle. In how many ways is it possible to split them into n pairs such that no segments connecting the two points in a pair intersect?

II A mountain profile consists of n upstrokes and n downstrokes, beginning and finishing at 'sea-level', such that the profile never gets lower than the 'sea-level'. How many such profiles are possible?

III A rooted tree is a tree (graph without cycles) where one node has been selected as the root (it is then considered different). How many rooted trees with n edges are there?

IV A rooted binary tree is a rooted tree where each node has either 0 or 2 children. How many rooted binary trees with $n + 1$ end nodes are possible?

V In how many ways can an $n+2$-gon be triangulated (split up into n triangles by $n - 1$ non-intersecting diagonals)?

VI In how many ways is it possible to arrange n sets of brackets so that none are closed before they are opened?

VII $n + 1$ numbers are written down in a given order with n multiplication symbols between them. In how many different ways can they be multiplied if n sets of brackets are used? (Eg. for $n = 2$ there are two: $((a \cdot b) \cdot c)$ and $(a \cdot (b \cdot c))$)?

Turn over for some hints

1. Find all possibilities for $n \leq 3$ by listing.

2. Prove that all of the problems are equivalent.

3. Prove that if c_n is the answer for n, then

$$c_{n+1} = \sum_{i=0}^{n} c_i c_{n-i}$$

4. Prove that

$$c_n = \frac{1}{n+1} \binom{2n}{n}$$