Helly's Theorem

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Def. **Convex set** - a set such that if it contains two points, it contains the whole segment joining the two points.

1. Prove that the intersection of two convex sets is convex.

2. Prove that if for four convex sets F_1, F_2, F_3, F_4 in the plane any three have non-empty intersection, all four have non-empty intersection.

Hint: consider four points a_1, a_2, a_3, a_4 such that $a_1 \in F_2 \cap F_3 \cap F_4$ etc.

Helly's Theorem If for a finite collection of convex sets in $\mathbb{R}^n$ any family of $n+1$ sets has non-empty intersection, then the whole collection has non-empty intersection.

3. Prove Helly's Theorem for $n = 2$ by induction on k, the number of sets in the collection.

Hint for the step: consider the sets $G_m = F_m \cap F_{k+1}$ for $1 \leq m \leq k$.

4. Given a finite collection of vertical segments in the plane, prove that there exists a horizontal line intersecting all the segments if for each three segments there is a horizontal line intersecting them.

5. Given a finite collection of vertical segments in the plane, prove that there exists a line intersecting all the segments if for each three segments there is a line intersecting them.

6. Let S be a set of points in the plane such that any three points can be covered by a disc of radius 1. Prove that S can be covered by a disc of radius 1.

7. Given n points in the plane, prove there exists a point P such that for any line through P there are at least $\frac{n}{3}$ points on each side of the line.