Inequalities

21 February 2018

The means

- AM - The arithmetic mean of $a_1, a_2, ..., a_n$ is $\frac{\sum_{i=1}^{n} a_i}{n}$
- GM - The geometric mean of $a_1, a_2, ..., a_n$ is $\sqrt[n]{\prod_{i=1}^{n} a_i}$
- HM - The harmonic mean of $a_1, a_2, ..., a_n$ is $\frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}}$
- RMS - The root mean square of $a_1, a_2, ..., a_n$ is $\sqrt{\frac{\sum_{i=1}^{n} a_i^2}{n}}$

Some inequalities

- The trivial inequality: for any real a, $a^2 \geq 0$
- The inequality of the means: for $a_1, a_2, ..., a_n > 0$, HM $\leq$ GM $\leq$ AM $\leq$ RMS
- The Cauchy-Schwarz inequality: if $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n$ are real numbers,

$$\sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2 \geq \left(\sum_{i=1}^{n} x_i y_i \right)^2$$

1. Prove the inequality of the means for $n = 2$ When do equalities hold?

2. Prove the Cauchy-Schwarz inequality for $n = 2$ When does equality hold?

3. What is the geometric meaning of the Cauchy-Schwarz inequality? (Hint: vectors)

4. * Prove AM-GM (ie that GM $\leq$ AM) for all n

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5. Prove that for all real x, y, z, w $\quad x^2 + y^2 + z^2 + w^2 + 1 \geq x + y + z + w$

6. Prove that if $a > 0$, then $1 + a \geq 2\sqrt{a}$

7. Prove that if $a, b, c > 0$, then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{a+b+c}$

8. Find the maximum volume of a cone inscribed in a sphere of radius r.

9. Prove that if $a, b, c > 0$, then $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}$

10. Find the maximum possible value of $x + 2y + 4z$ given that $x^2 + y^2 + 2z^2 = 1$

11. Let $p(x)$ be a polynomial with positive coefficients. Prove that for $x > 0$, $p(\frac{1}{x}) \geq \frac{1}{p(x)}$