The Rearrangement Inequality

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Rearrangement Inequality Let (a_k) and (b_k) be finite sequences of real numbers of length n such that $a_1 \leq a_2 \leq ... \leq a_n$, $b_1 \leq b_2 \leq ... \leq b_n$. Let (B_k) be a permutation of (b_k). Then

$$a_1b_1 + a_2b_2 + ... + a_nb_n \geq a_1B_1 + a_2B_2 + ... + a_nB_n \geq a_1b_n + a_2b_{n-1} + ... + a_nb_1.$$

1. * Prove the rearrangement inequality.

2. Use the rearrangement inequality to prove AM-GM.

3. Use the rearrangement inequality to prove Cauchy-Schwarz.

4. Let $x_i, y_i (i = 1, 2, ..., n)$ be real numbers such that

$$x_1 \geq x_2 \geq \cdots \geq x_n \text{ and } y_1 \geq y_2 \geq \cdots \geq y_n.$$

Prove that if $z_1, z_2, \cdots, z_n$ is any permutation of $y_1, y_2, \cdots, y_n$, then

$$\sum_{i=1}^{n} (x_i - y_i)^2 \leq \sum_{i=1}^{n} (x_i - z_i)^2.$$

5. Prove that for $a, b, c > 0$,

$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2}.$$

6. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$