Inversion: Harder problems

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1.   Given the circle of inversion and a point A construct its inverse A'.

2.   Given two non-intersecting circles Γ_1, Γ_2 and a point P outside the circles, how many circles through P are there that are tangent to Γ_1 and Γ_2?
(*) Construct any of them.

3.   Find the locus of the point of tangency of pairs of tangent circles inscribed in a segment.

4. Steiner Chains   Prove that any two circles can be mapped to two concentric ones by a composition of inversions. Hence prove the following result:

Circle Γ_1 lies inside Γ_2. A circle ω_0 is chosen tangent to Γ_1 (externally) and Γ_2. A chain of circles (ω_n) is formed where circle ω_{k+1} is the circle tangent to ω_k, Γ_1 and Γ_2. Sometimes the last circle ω_n is tangent to ω_0, the chain is then called closed. Prove that if there is a circle ω_0 for which the chain is closed, then the chain starting with any circle S tangent to Γ_1 and Γ_2 is closed.

5. Feuerbach's Theorem   Prove that the circle through the midpoints of the sides of a triangle is tangent to the incircle and the excircles of the triangle. (The circle is called the nine-point circle and has a number of remarkable properties).