Beatty's Theorem via Cutting Sequences

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April 10th 2020

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For an irrational number $\alpha > 0$, its **Beatty sequence** $\mathcal{B}^\alpha$ is given by $\mathcal{B}_n^\alpha = \lfloor n\alpha \rfloor$.

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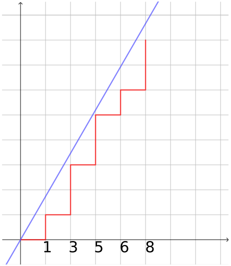

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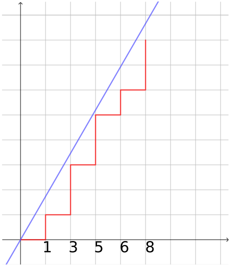

Beatty's Theorem (Rayleigh 1894, Beatty 1926)

The Beatty sequences of α and β partition the positive integers if and only if $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

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For a line ℓ with irrational slope $\alpha > 0$, its cutting sequence $\mathcal{C}^\alpha$ is obtained by writing a 0 for each vertical grid line crossed by ℓ, and a 1 for each horizontal grid line crossed.

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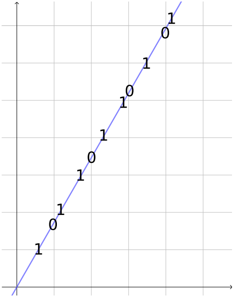

The cutting sequence for $\sqrt{3}$ is 10110110101...

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$\mathcal{C}^{\sqrt{3}}$	1	0	1	1	0	1	1	0	1	0 ...
$\mathcal{B}^{\sqrt{3}}$		1			3			5		6 ...

Proof of Beatty's Theorem

First note that

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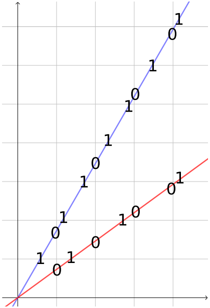

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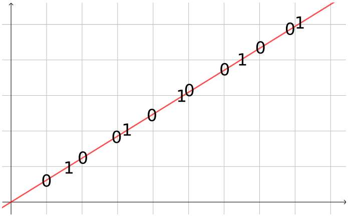
And so

$$\begin{aligned} \mathcal{B}^\alpha, \mathcal{B}^\beta \text{ partition } \mathbb{N} &\iff \mathcal{C}^{\alpha-1} \text{ and } \mathcal{C}^{\beta-1} \text{ are inverse} \\ &\iff (\alpha-1)(\beta-1) = 1 \\ &\iff \frac{1}{\alpha} + \frac{1}{\beta} = 1 \end{aligned}$$

□

Cutting sequence for $\frac{1}{\varphi}$



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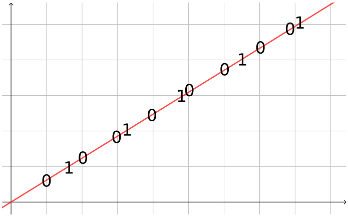

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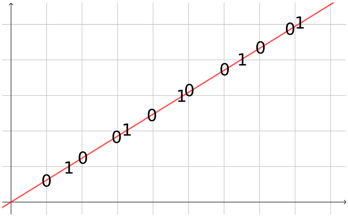

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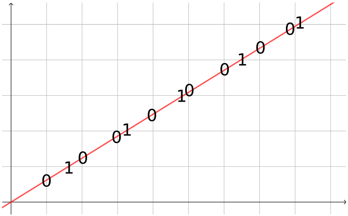

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So $\mathcal{C}^{\frac{1}{\varphi}}$ is invariant under:

$$0 \mapsto 01$$

$$1 \mapsto 0$$

The Fibonacci Word

The 0th Fibonacci word, S_0, is 0.
S_n is obtained from S_{n-1} by the substitution

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By induction, $S_{n+1} = S_n S_{n-1}$, and S_n contains F_n 1s, F_{n+1} 0s.

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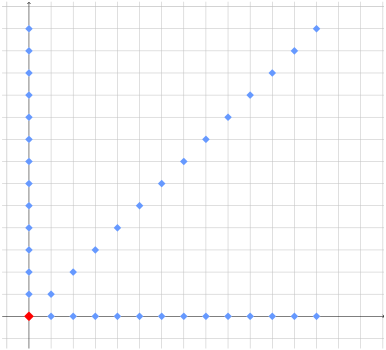

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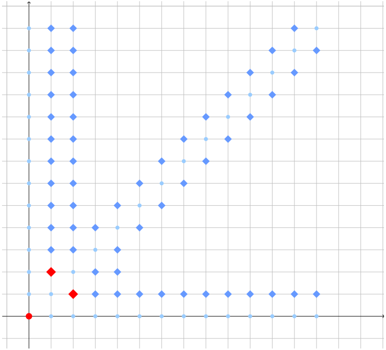

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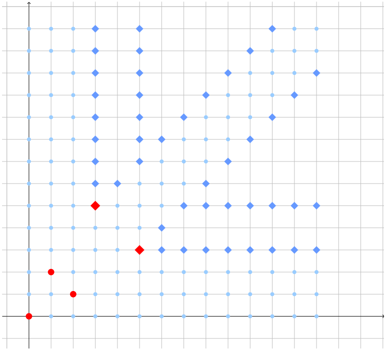

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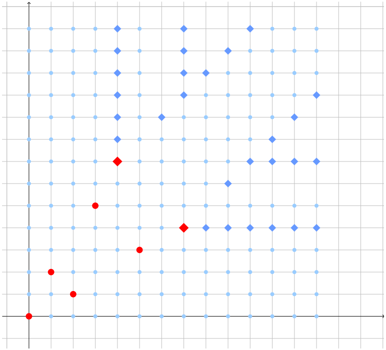

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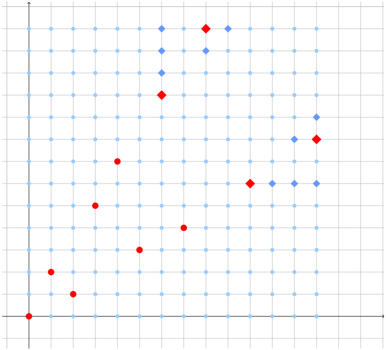

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We can also construct the sequences as follows:

- L_n is the lowest positive integer that hasn't appeared in either (L_n) or (U_n) yet.
- $U_n = L_n + n$.

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So we can construct te Beatty sequences in the same way as the Wythoff sequences:

- $\mathcal{B}_n^{\varphi}$ is the lowest positive integer that hasn't appeared in either $\mathcal{B}^{\varphi}$ or $\mathcal{B}^{\varphi^2}$ yet.
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$$L_n = \mathcal{B}_n^{\varphi} = \lfloor n\varphi \rfloor$$
$$U_n = \mathcal{B}_n^{\varphi^2} = \lfloor n\varphi^2 \rfloor.$$

Bibliography

Serge Tabachnikov, *Geometry and Billiards*, American Mathematical Society, 2005.