What is . . . a translation surface?

Yuriy Tumarkin

Zurich Graduate Colloquium
16th April 2024

Plan

1. What is a translation surface?

Plan

1. What is a translation surface?
2. Where do they live?

Plan

1. What is a translation surface?
2. Where do they live? (On a moduli space)

Plan

1. What is a translation surface?
2. Where do they live? (On a moduli space)
3. What can you do with it?

Plan

1. What is a translation surface?
2. Where do they live? (On a moduli space)
3. What can you do with it? (Dynamics)

Plan

1. What is a translation surface?
2. Where do they live? (On a moduli space)
3. What can you do with it? (Dynamics)
4. How do you study the dynamics?

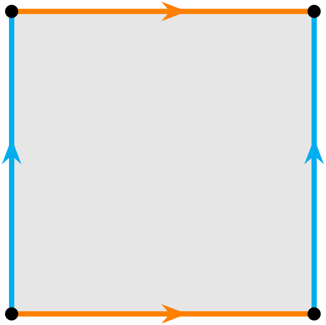
Plan

1. What is a translation surface?
2. Where do they live? (On a moduli space)
3. What can you do with it? (Dynamics)
4. How do you study the dynamics? (By renormalisation)

What is a translation surface?

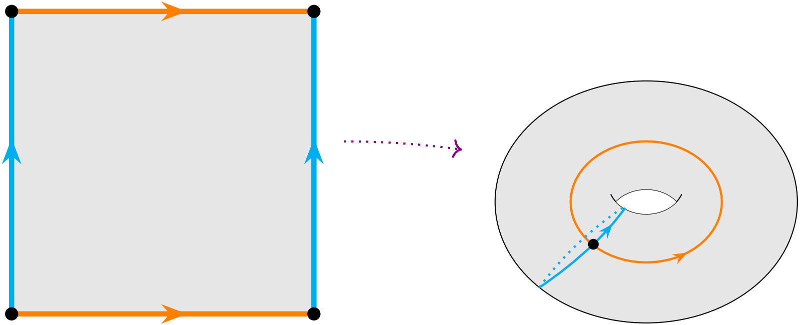
First example: the flat torus

Glue the opposite sides of a square:



First example: the flat torus

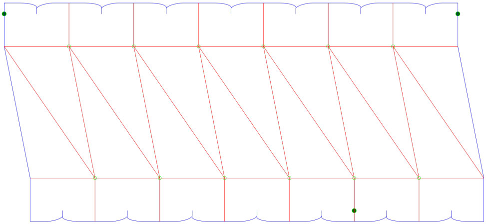
Glue the opposite sides of a square:

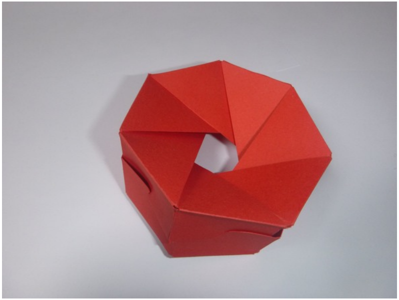


Aside: diplotori

Alba Málaga, Samuel Lelièvre, Pierre Arnoux

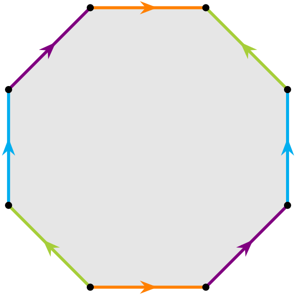
The usual embedding of a torus in $\mathbb{R}^3$ is not flat, but it is in fact possible to fold a torus out of paper:





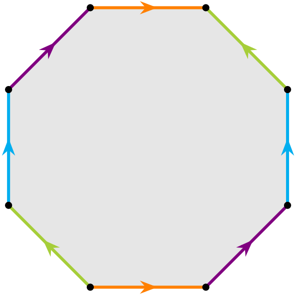
Higher genus translation surfaces

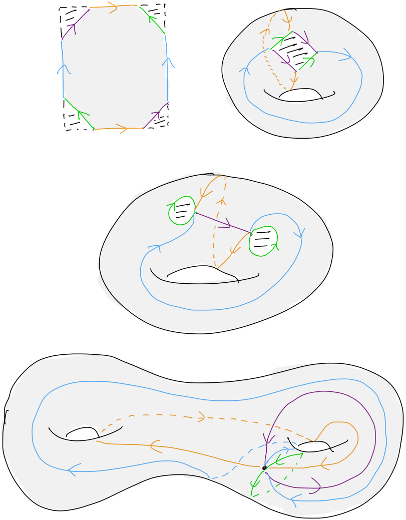
Now suppose we start with a regular octagon:



Higher genus translation surfaces

Now suppose we start with a regular octagon:



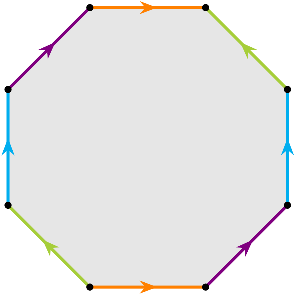


We get a genus 2 surface:

Cone singularities

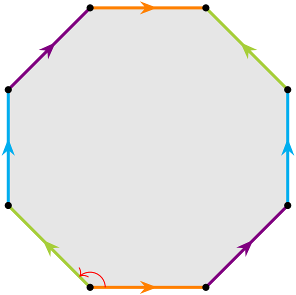
Cone singularities

Trace a small loop around the image of one vertex of the octagon:



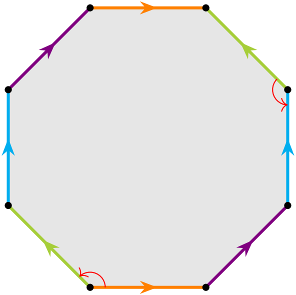
Cone singularities

Trace a small loop around the image of one vertex of the octagon:



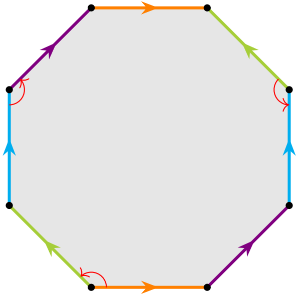
Cone singularities

Trace a small loop around the image of one vertex of the octagon:



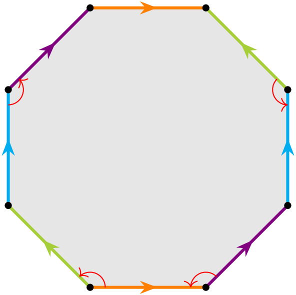
Cone singularities

Trace a small loop around the image of one vertex of the octagon:



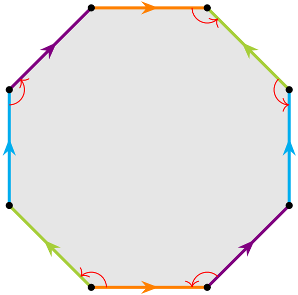
Cone singularities

Trace a small loop around the image of one vertex of the octagon:



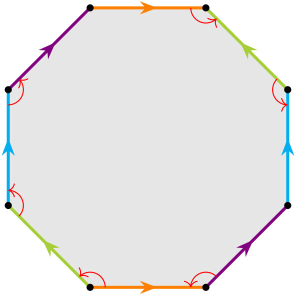
Cone singularities

Trace a small loop around the image of one vertex of the octagon:



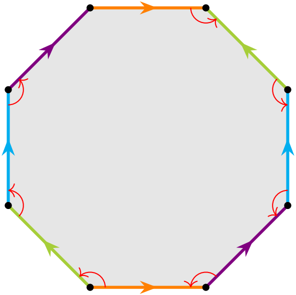
Cone singularities

Trace a small loop around the image of one vertex of the octagon:



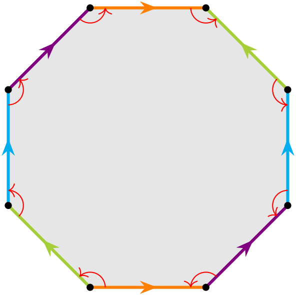
Cone singularities

Trace a small loop around the image of one vertex of the octagon:



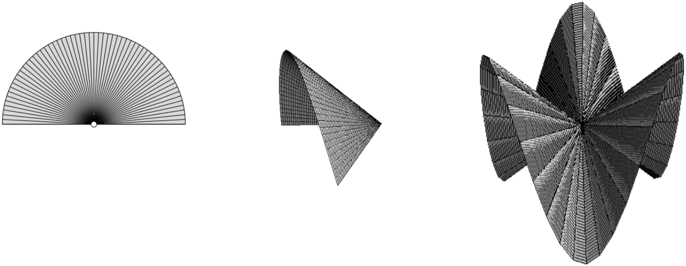
Cone singularities

Trace a small loop around the image of one vertex of the octagon:



Cone singularities

All the vertices are glued together, so the resulting angle there is $8 \times \frac{3}{4}\pi = 6\pi$.



[Picture: Zorich]

Definition of translation surface

Definition

A **translation surface** is a space obtained by identifying pairwise all the edges of a collection of polygons $\{P_1, P_2, \dots\}$ in $\mathbb{R}^2$, where for each pair (a_i, b_i) of identified edges,

- a_i and b_i are parallel and have the same length.
- a_i and b_i are on opposite sides of their respective polygons (where the boundaries of the P_i are all oriented counter-clockwise).

Definition of translation surface

Definition

A **translation surface** is a space obtained by identifying pairwise all the edges of a collection of polygons $\{P_1, P_2, \dots\}$ in $\mathbb{R}^2$, where for each pair (a_i, b_i) of identified edges,

- a_i and b_i are parallel and have the same length.
- a_i and b_i are on opposite sides of their respective polygons (where the boundaries of the P_i are all oriented counter-clockwise).

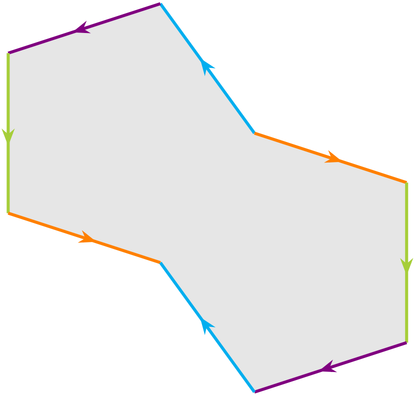
Removing the singularities, a translation surface is a surface with charts such that all transition functions are translations - hence the name.

The cone angle at each singularity is always an integer multiple of 2π.

Examples

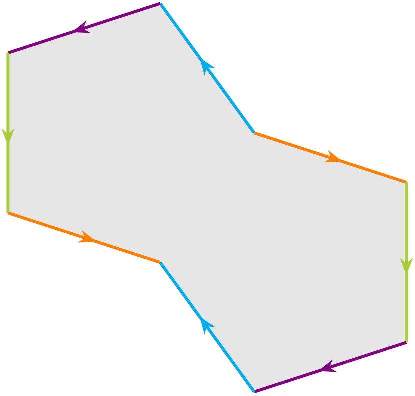
Examples

Double pentagon

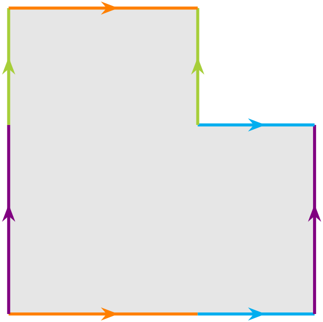


Examples

Double pentagon

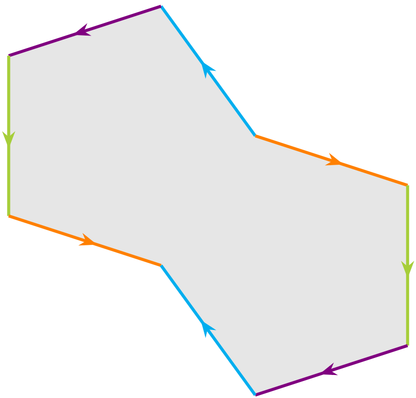


Golden L

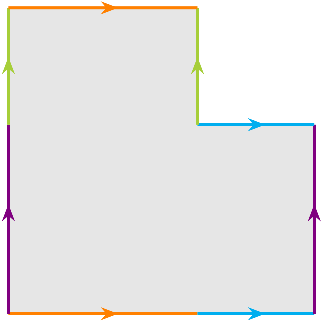


Examples

Double pentagon



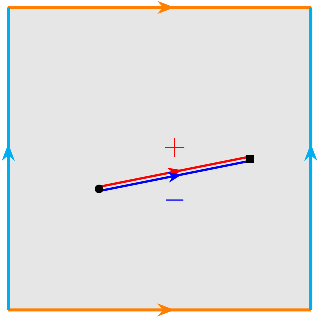
Golden L

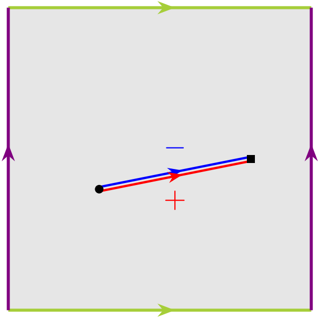


Both surfaces have genus 2, one singularity with angle 6π – same topology as the regular octagon.

More examples

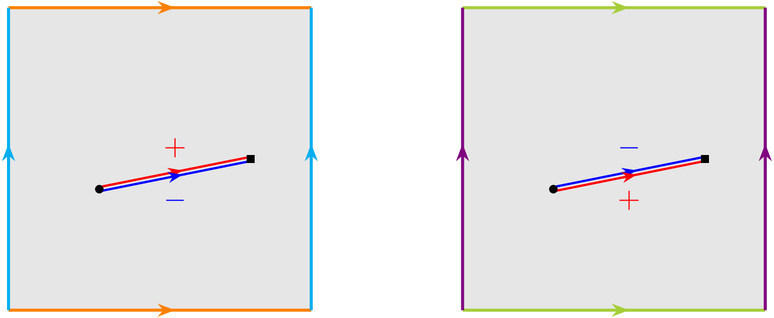
Doubled slit torus





More examples

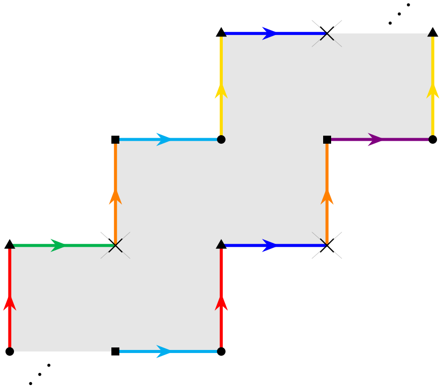
Doubled slit torus



Also genus 2, but with two singularities, each of angle 4π.

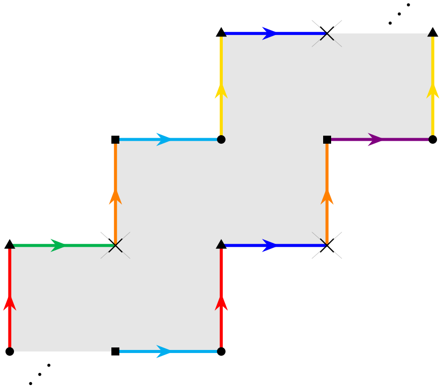
More examples

Infinite staircase



More examples

Infinite staircase

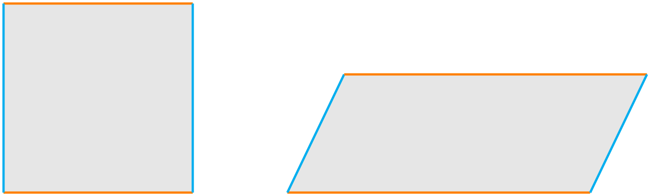


Infinite genus, four **wild** singularities (infinite cone angle).

Where do translation surfaces live?

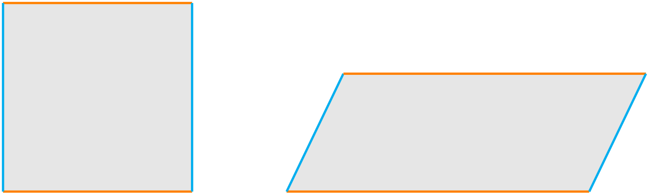
Different shapes of tori

The following two tori are topologically the same, but have different metrics:



Different shapes of tori

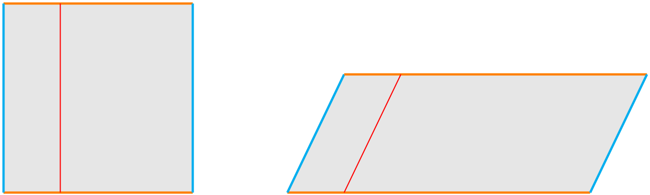
The following two tori are topologically the same, but have different metrics:



How do we know they're different? The shortest closed curve is shorter on the right torus.

Different shapes of tori

The following two tori are topologically the same, but have different metrics:



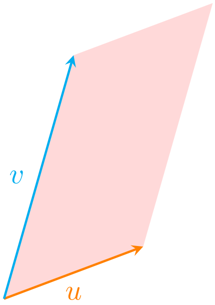
How do we know they're different? The shortest closed curve is shorter on the right torus.

How many different tori are there?

Want to classify flat tori up to scaling and rotation.

How many different tori are there?

Want to classify flat tori up to scaling and rotation.
A torus is defined by two vectors $u, v \in \mathbb{C}$ spanning the parallelogram.

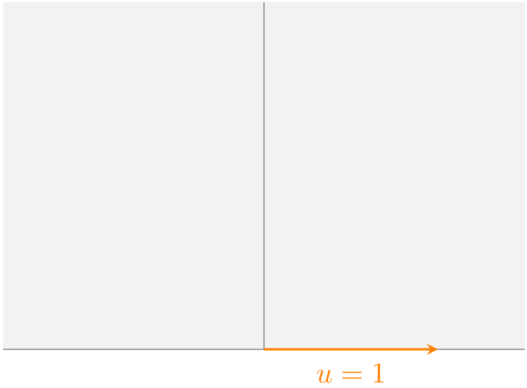


How many different tori are there?

Want to classify flat tori up to scaling and rotation.

A torus is defined by two vectors $u, v \in \mathbb{C}$ spanning the parallelogram.

- Up to scaling and rotation can assume $u = 1$.

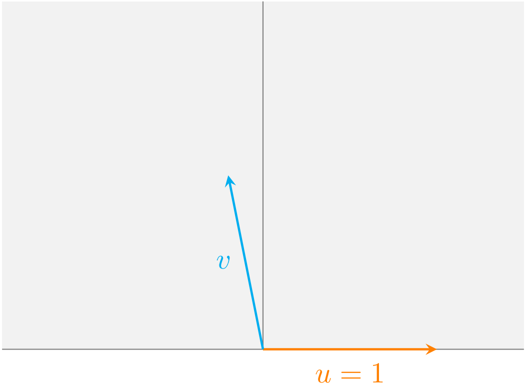


How many different tori are there?

Want to classify flat tori up to scaling and rotation.

A torus is defined by two vectors $u, v \in \mathbb{C}$ spanning the parallelogram.

- Up to scaling and rotation can assume $u = 1$.
- Then v can be anywhere in the upper half-plane.



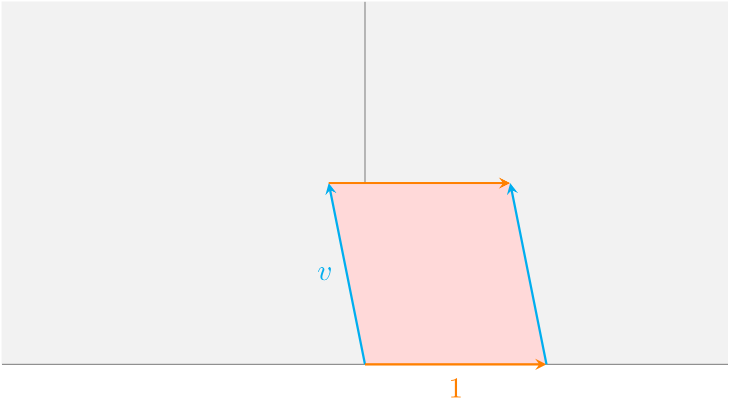
Equivalent tori

What choices of v give the same torus?

Equivalent tori

What choices of v give the same torus?

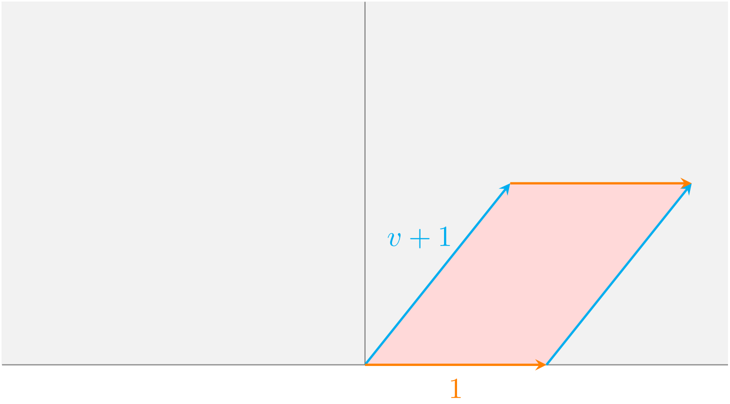
1. v and $v+1$:



Equivalent tori

What choices of v give the same torus?

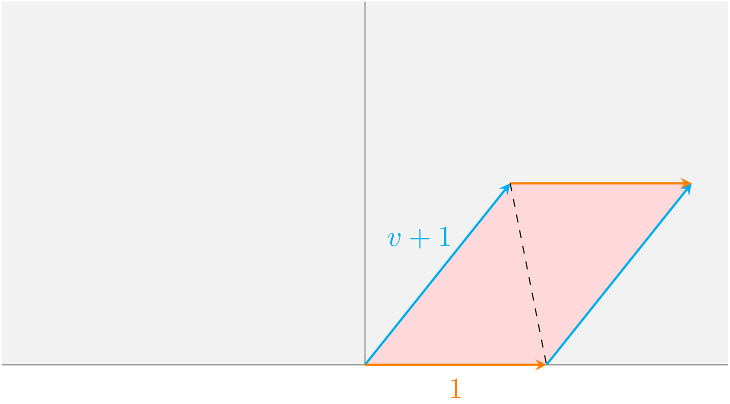
1. v and $v + 1$:



Equivalent tori

What choices of v give the same torus?

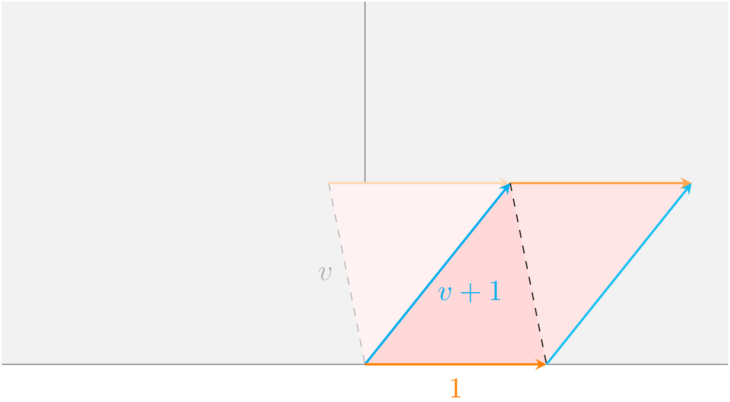
1. v and $v + 1$:



Equivalent tori

What choices of v give the same torus?

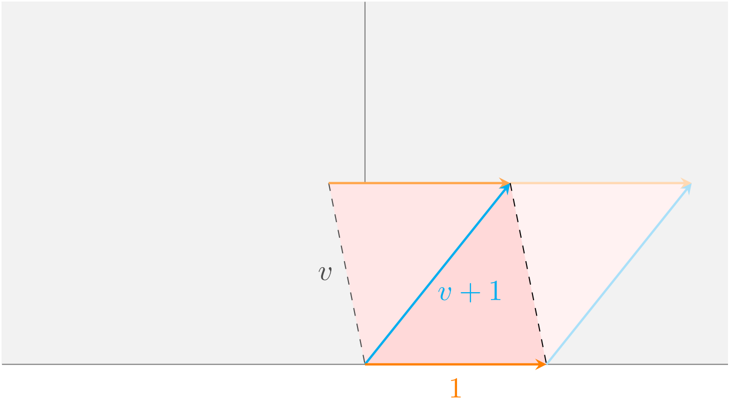
1. v and $v + 1$:



Equivalent tori

What choices of v give the same torus?

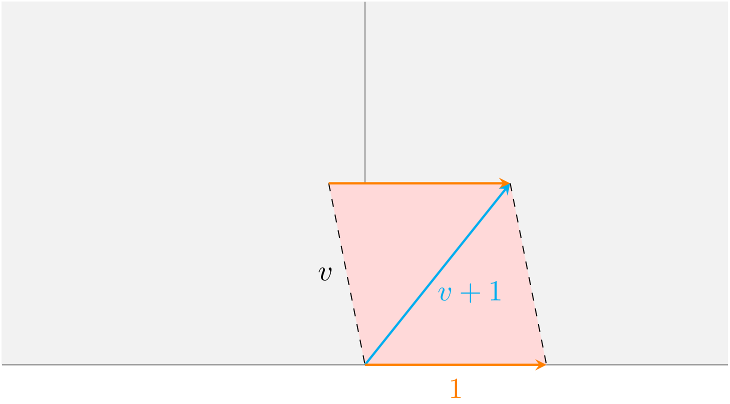
1. v and $v+1$:



Equivalent tori

What choices of v give the same torus?

1. v and $v+1$:



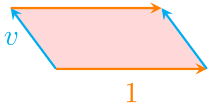
Equivalent tori

What choices of v give the same torus?

Equivalent tori

What choices of v give the same torus?

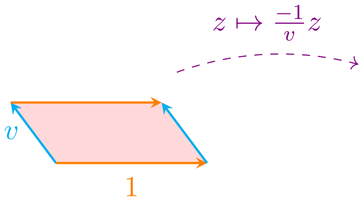
2. v and $\frac{-1}{v}$:



Equivalent tori

What choices of v give the same torus?

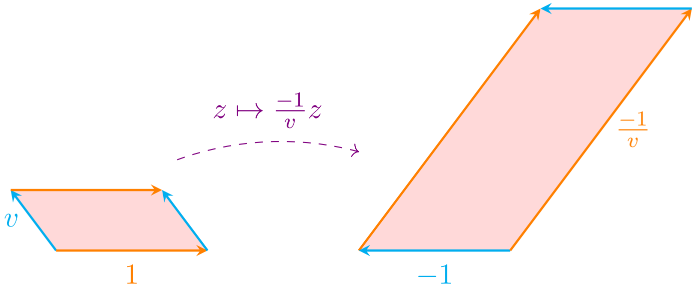
2. v and $\frac{-1}{v}$:



Equivalent tori

What choices of v give the same torus?

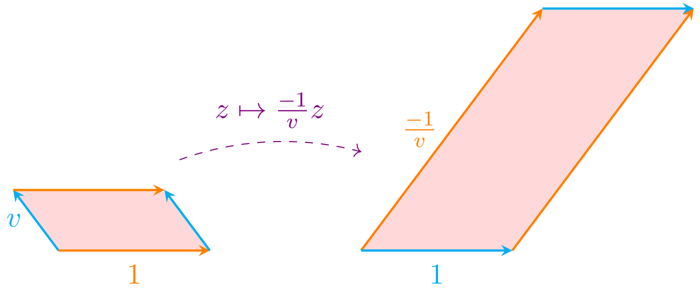
2. v and $\frac{-1}{v}$:



Equivalent tori

What choices of v give the same torus?

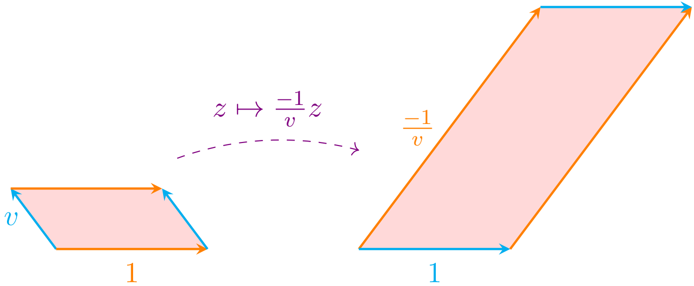
2. v and $\frac{-1}{v}$:



Equivalent tori

What choices of v give the same torus?

2. v and $\frac{-1}{v}$:



Up to rotation and scaling, same torus.

Moduli space of tori

- Using (1): $v \sim v+1$, can always pick v such that $|\mathrm{Re}(v)| \leq \frac{1}{2}$.

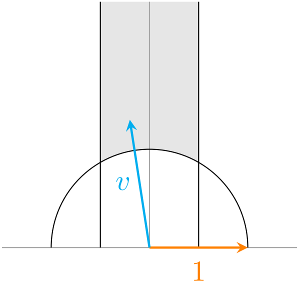
Moduli space of tori

- Using (1): $v \sim v + 1$, can always pick v such that $|\mathrm{Re}(v)| \leq \frac{1}{2}$.
- Using (2): $v \sim \frac{-1}{v}$ can assume u is shorter than v, i.e. $|v| \geq 1$.

Moduli space of tori

- Using (1): $v \sim v + 1$, can always pick v such that $|\mathrm{Re}(v)| \leq \frac{1}{2}$.
- Using (2): $v \sim \frac{-1}{v}$ can assume u is shorter than v, i.e. $|v| \geq 1$.

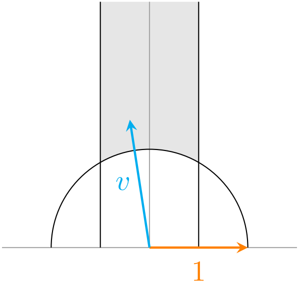
Thus: any torus can be uniquely given by a choice of v in the shaded region, the **moduli space** of flat tori:



Moduli space of tori

- Using (1): $v \sim v+1$, can always pick v such that $|\mathrm{Re}(v)| \leq \frac{1}{2}$.
- Using (2): $v \sim \frac{-1}{v}$ can assume u is shorter than v, i.e. $|v| \geq 1$.

Thus: any torus can be uniquely given by a choice of v in the shaded region, the **moduli space** of flat tori:



There is a similar moduli space for translation surfaces of any genus.

Teichmüller geodesic flow

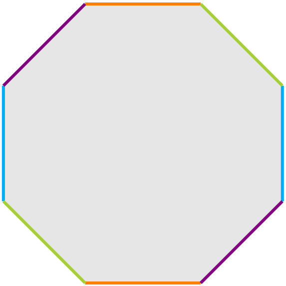
Definition

The **Teichmüller geodesic flow** is the flow φ_t on the moduli space of translation surfaces, where $\varphi_t(M)$ is obtained by stretching M in the x-direction by a factor of e^t, and contracting in the y-direction by the same factor.

Teichmüller geodesic flow

Definition

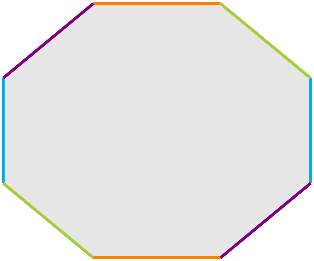
The **Teichmüller geodesic flow** is the flow φ_t on the moduli space of translation surfaces, where $\varphi_t(M)$ is obtained by stretching M in the x-direction by a factor of e^t, and contracting in the y-direction by the same factor.



Teichmüller geodesic flow

Definition

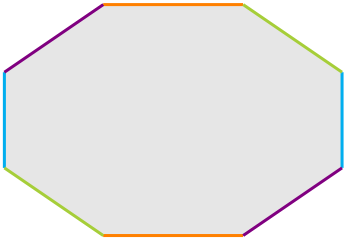
The **Teichmüller geodesic flow** is the flow φ_t on the moduli space of translation surfaces, where $\varphi_t(M)$ is obtained by stretching M in the x-direction by a factor of e^t, and contracting in the y-direction by the same factor.



Teichmüller geodesic flow

Definition

The **Teichmüller geodesic flow** is the flow φ_t on the moduli space of translation surfaces, where $\varphi_t(M)$ is obtained by stretching M in the x-direction by a factor of e^t, and contracting in the y-direction by the same factor.



Teichmüller geodesic flow

Definition

The **Teichmüller geodesic flow** is the flow φ_t on the moduli space of translation surfaces, where $\varphi_t(M)$ is obtained by stretching M in the x-direction by a factor of e^t, and contracting in the y-direction by the same factor.

What can you do with a translation surface?

Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

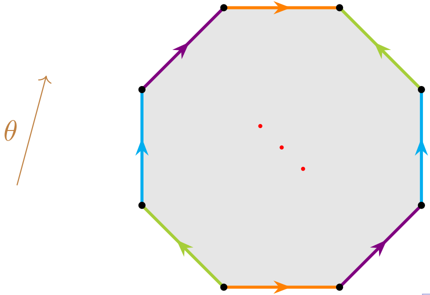
The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

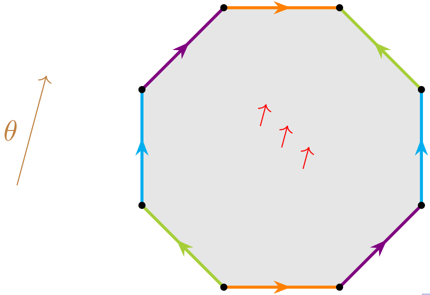


Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

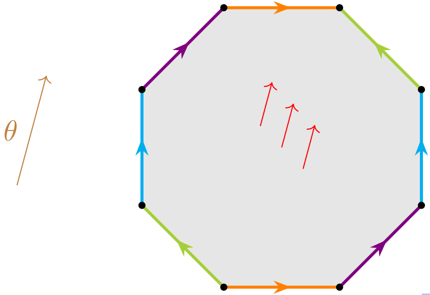


Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

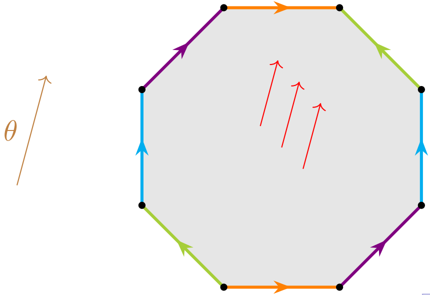


Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

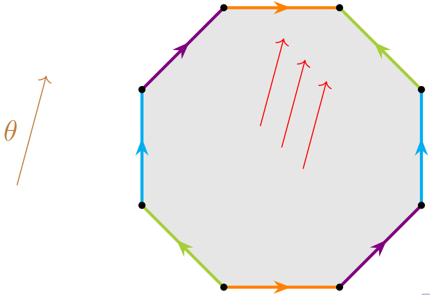


Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

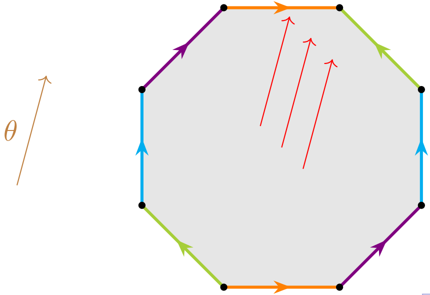


Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

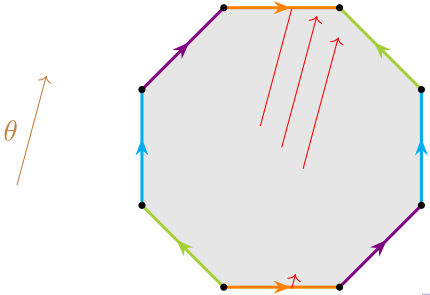


Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

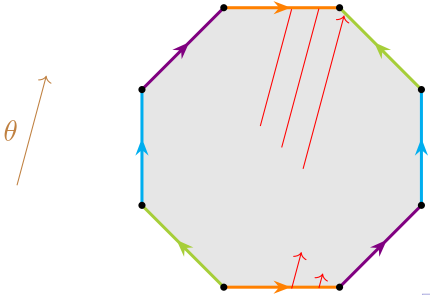


Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

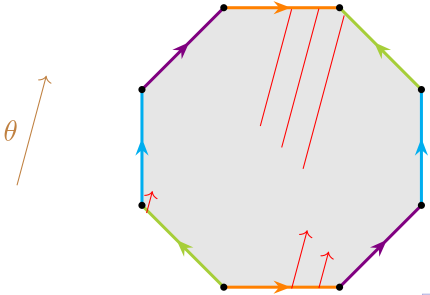


Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

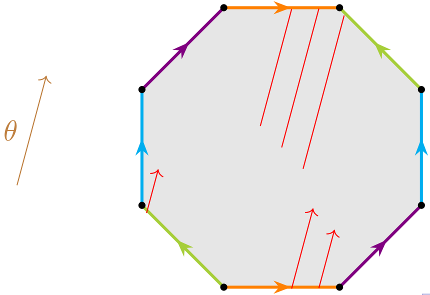


Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

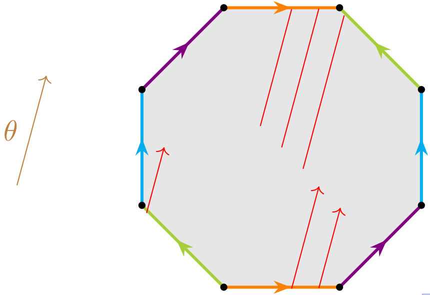


Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

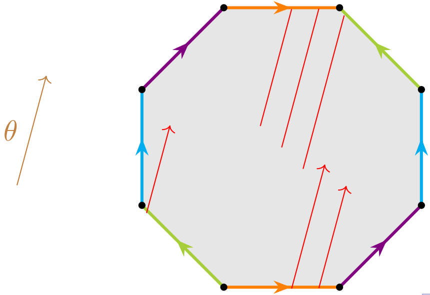


Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.

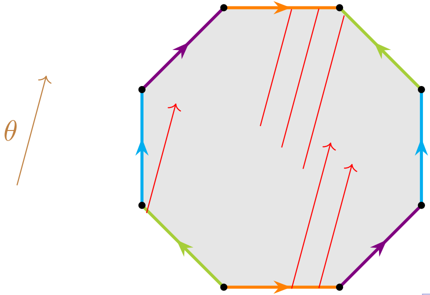


Straight-line flow

Always glue by translations $\implies$ well-defined notion of direction: can put a compass everywhere on the surface and it will show which way is north.

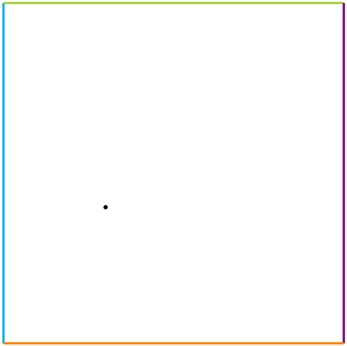
Definition

The **straight-line flow** in direction $\theta \in S^1$ is the flow in direction θ at unit speed.



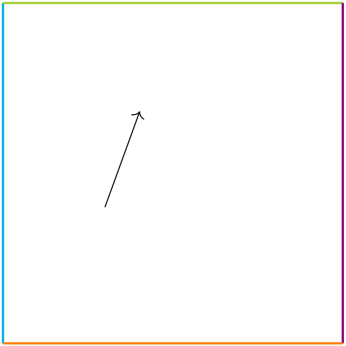
Translation surfaces from billiards

Consider the billiard in a square:



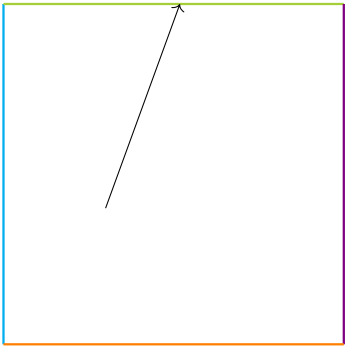
Translation surfaces from billiards

Consider the billiard in a square:



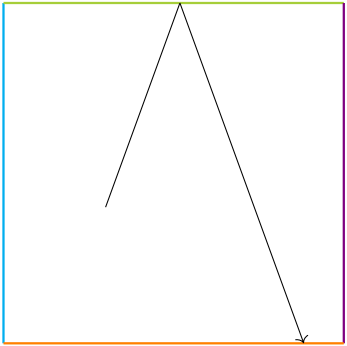
Translation surfaces from billiards

Consider the billiard in a square:



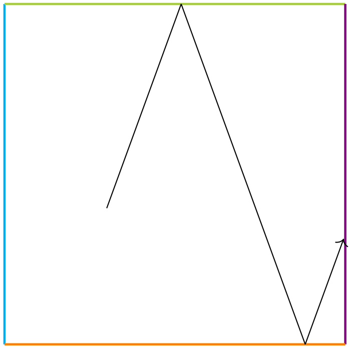
Translation surfaces from billiards

Consider the billiard in a square:



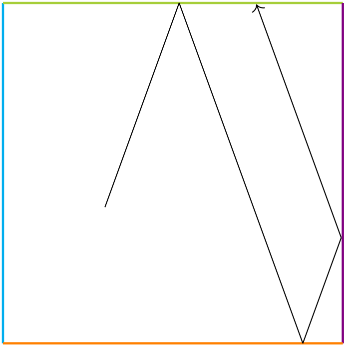
Translation surfaces from billiards

Consider the billiard in a square:



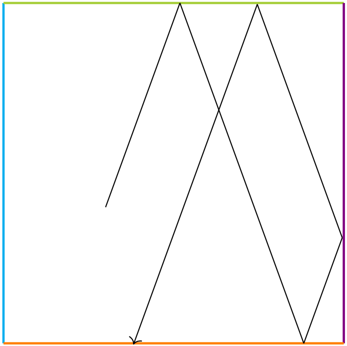
Translation surfaces from billiards

Consider the billiard in a square:



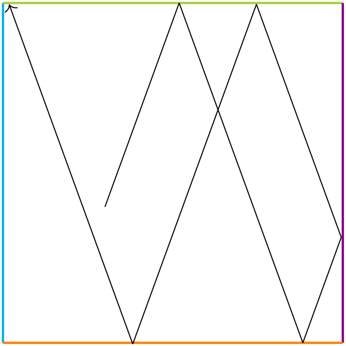
Translation surfaces from billiards

Consider the billiard in a square:



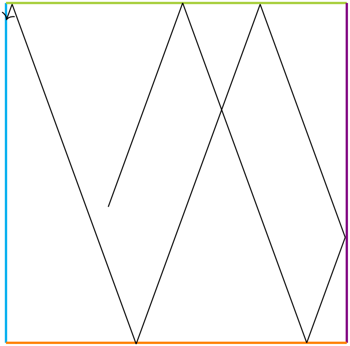
Translation surfaces from billiards

Consider the billiard in a square:



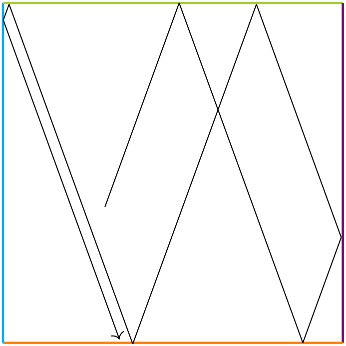
Translation surfaces from billiards

Consider the billiard in a square:



Translation surfaces from billiards

Consider the billiard in a square:



Translation surfaces from billiards

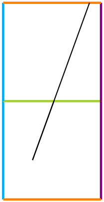
Instead of reflecting the trajectory, reflect the billiard table:

Translation surfaces from billiards

Instead of reflecting the trajectory, reflect the billiard table:

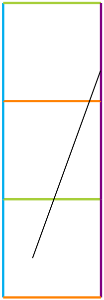
Translation surfaces from billiards

Instead of reflecting the trajectory, reflect the billiard table:



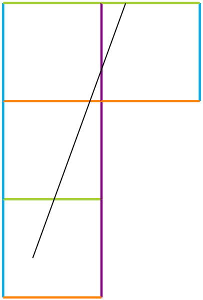
Translation surfaces from billiards

Instead of reflecting the trajectory, reflect the billiard table:



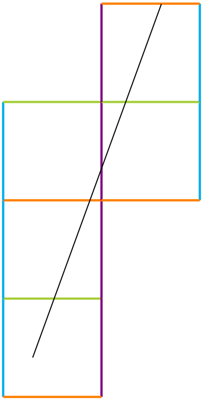
Translation surfaces from billiards

Instead of reflecting the trajectory, reflect the billiard table:



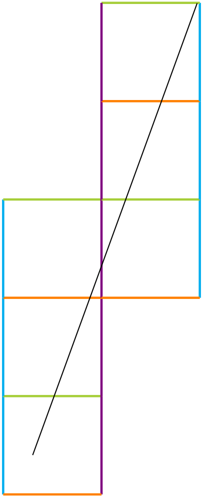
Translation surfaces from billiards

Instead of reflecting the trajectory, reflect the billiard table:



Translation surfaces from billiards

Instead of reflecting the trajectory, reflect the billiard table:

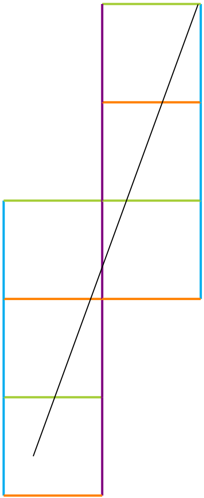


Translation surfaces from billiards

Now identify the squares with the same orientation:

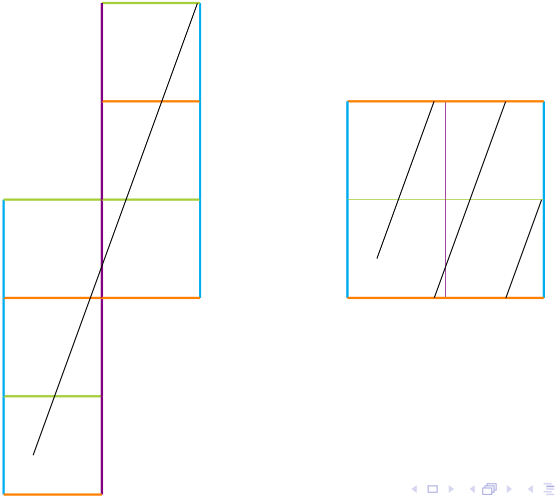
Translation surfaces from billiards

Now identify the squares with the same orientation:



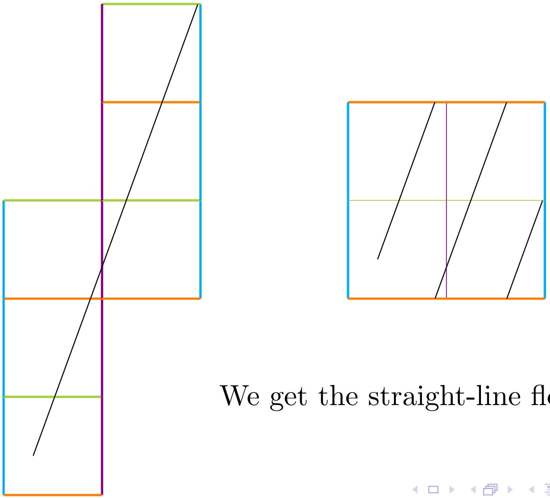
Translation surfaces from billiards

Now identify the squares with the same orientation:



Translation surfaces from billiards

Now identify the squares with the same orientation:



We get the straight-line flow on a torus!

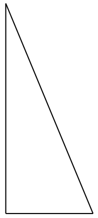
Translation surfaces from billiards: genus 2 example

Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

Translation surfaces from billiards: genus 2 example

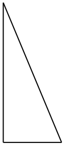
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

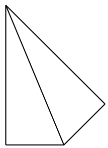
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

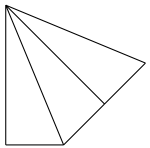
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

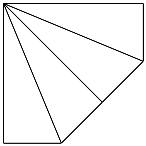
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

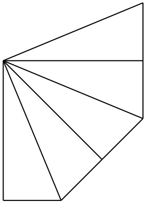
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

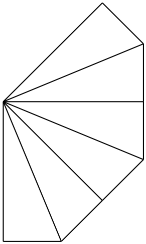
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

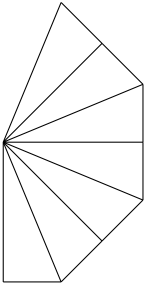
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

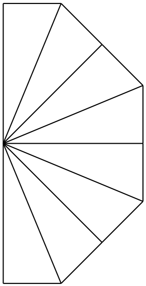
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

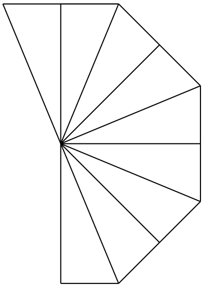
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

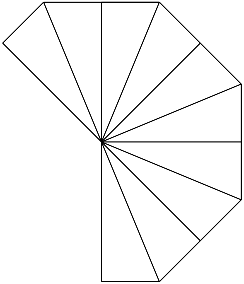
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

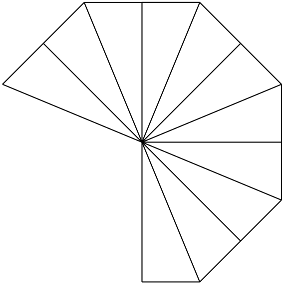
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

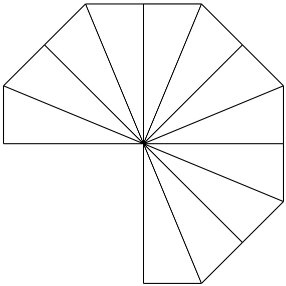
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

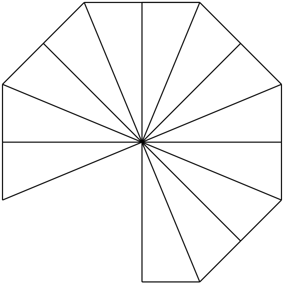
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

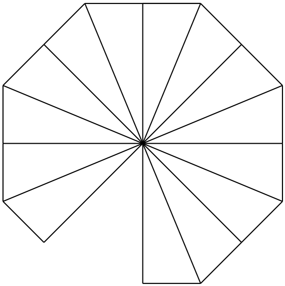
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



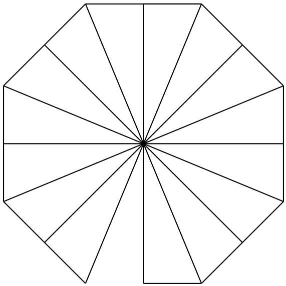
Translation surfaces from billiards: genus 2 example

Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



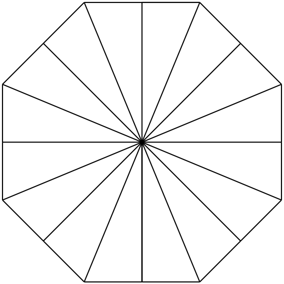
Translation surfaces from billiards: genus 2 example

Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

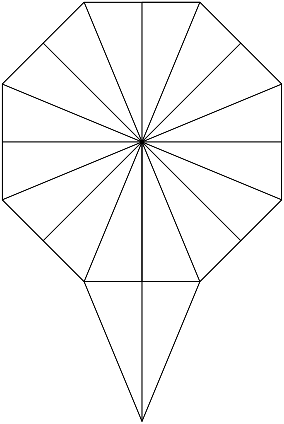
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

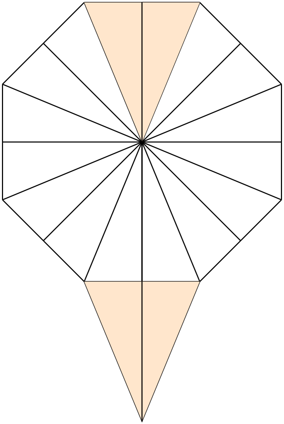
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

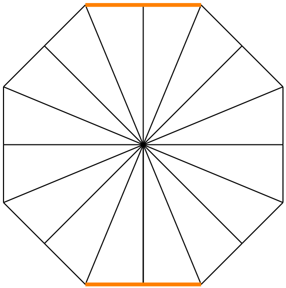
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

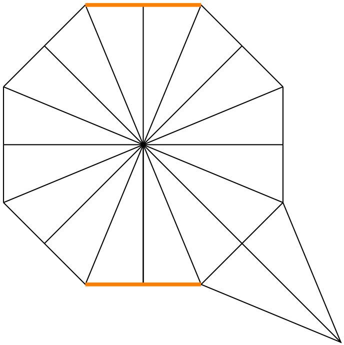
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

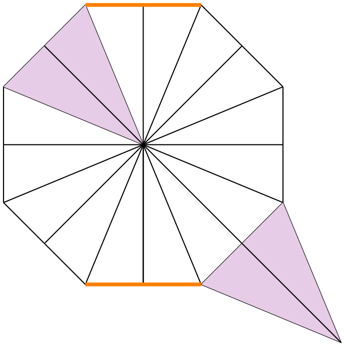
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



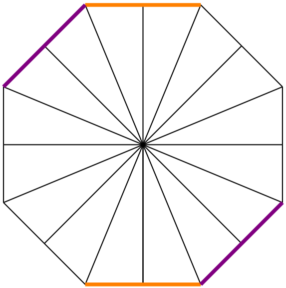
Translation surfaces from billiards: genus 2 example

Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

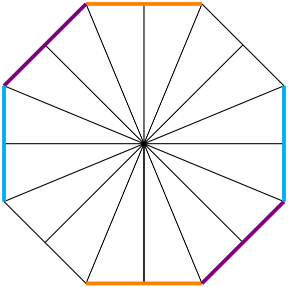
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

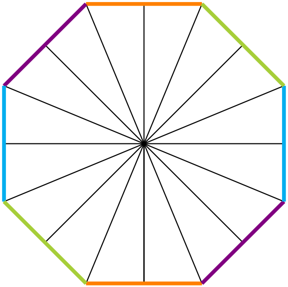
Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)

Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



Translation surfaces from billiards: genus 2 example

Can do this for any billiard table *with rational angles*. (Katok – Zemlyakov construction)
Example for the octagon: triangle with angles $\frac{\pi}{2}, \frac{\pi}{8}, \frac{3\pi}{8}$



How do you study the dynamics?

Ergodicity

To understand a dynamical system $T : X \to X$, the first question is: what does it preserve?

Ergodicity

To understand a dynamical system $T : X \to X$, the first question is: what does it preserve?

1. Are there any fixed points?

Ergodicity

To understand a dynamical system $T : X \to X$, the first question is: what does it preserve?

1. Are there any fixed points? For straight-line flow: no.

Ergodicity

To understand a dynamical system $T : X \to X$, the first question is: what does it preserve?

1. Are there any fixed points? For straight-line flow: no.
2. Are there any periodic orbits?

Ergodicity

To understand a dynamical system $T : X \to X$, the first question is: what does it preserve?

1. Are there any fixed points? For straight-line flow: no.
2. Are there any periodic orbits? For straight-line flow: usually no.

Ergodicity

To understand a dynamical system $T : X \to X$, the first question is: what does it preserve?

1. Are there any fixed points? For straight-line flow: no.
2. Are there any periodic orbits? For straight-line flow: usually no.
3. What are the invariant sets?

Ergodicity

To understand a dynamical system $T : X \to X$, the first question is: what does it preserve?

1. Are there any fixed points? For straight-line flow: no.
2. Are there any periodic orbits? For straight-line flow: usually no.
3. What are the invariant sets?

Definition

Let $T : X \to X$ be a dynamical system. A measure μ on X is **invariant** if for any measurable set $B \subset X$, $\mu(T^{-1}(X)) = \mu(X)$.

Ergodicity

To understand a dynamical system $T : X \to X$, the first question is: what does it preserve?

1. Are there any fixed points? For straight-line flow: no.
2. Are there any periodic orbits? For straight-line flow: usually no.
3. What are the invariant sets?

Definition

Let $T : X \to X$ be a dynamical system. A measure μ on X is **invariant** if for any measurable set $B \subset X$, $\mu(T^{-1}(X)) = \mu(X)$. An invariant measure μ is **ergodic** if all invariant sets B have $\mu(B) = 0$ or $\mu(X \setminus B) = 0$.

Ergodicity

To understand a dynamical system $T : X \to X$, the first question is: what does it preserve?

1. Are there any fixed points? For straight-line flow: no.
2. Are there any periodic orbits? For straight-line flow: usually no.
3. What are the invariant sets?

Definition

Let $T : X \to X$ be a dynamical system. A measure μ on X is **invariant** if for any measurable set $B \subset X$, $\mu(T^{-1}(X)) = \mu(X)$. An invariant measure μ is **ergodic** if all invariant sets B have $\mu(B) = 0$ or $\mu(X \setminus B) = 0$.

Ergodicity means "as seen by μ, the dynamics doesn't decompose".

Unique ergodicity

A slightly finer question than finding invariant sets is:

4. What are the invariant measures?

Unique ergodicity

A slightly finer question than finding invariant sets is:

4. What are the invariant measures?

Definition

$T : X \to X$ is **uniquely ergodic** if it has only one invariant measure.

Unique ergodicity

A slightly finer question than finding invariant sets is:

4. What are the invariant measures?

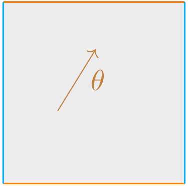
Definition

$T : X \to X$ is **uniquely ergodic** if it has only one invariant measure.

Unique ergodicity means “the dynamics does not decompose”.

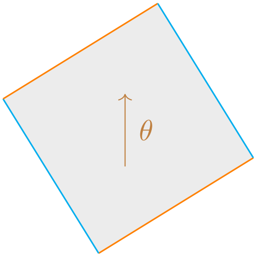
Unique ergodicity for periodic type

Consider the straight-line flow on a translation surface in direction θ.



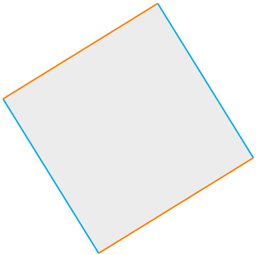
Unique ergodicity for periodic type

Consider the straight-line flow on a translation surface in direction θ. We can rotate the surface so that the flow is vertically upwards.



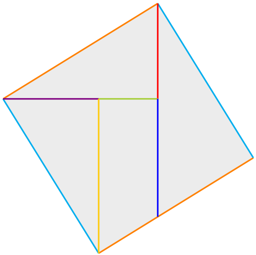
Unique ergodicity for periodic type

Consider the straight-line flow on a translation surface in direction θ. We can rotate the surface so that the flow is vertically upwards. Cut and paste to make the surface out of rectangles:



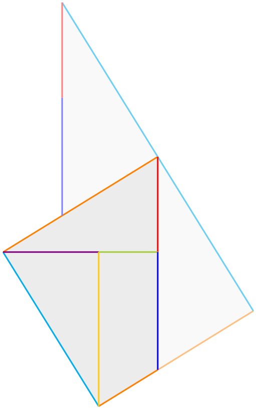
Unique ergodicity for periodic type

Consider the straight-line flow on a translation surface in direction θ. We can rotate the surface so that the flow is vertically upwards. Cut and paste to make the surface out of rectangles:



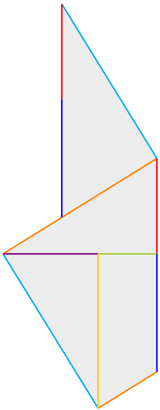
Unique ergodicity for periodic type

Consider the straight-line flow on a translation surface in direction θ. We can rotate the surface so that the flow is vertically upwards. Cut and paste to make the surface out of rectangles:



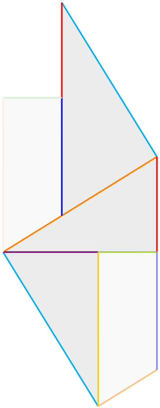
Unique ergodicity for periodic type

Consider the straight-line flow on a translation surface in direction θ. We can rotate the surface so that the flow is vertically upwards. Cut and paste to make the surface out of rectangles:



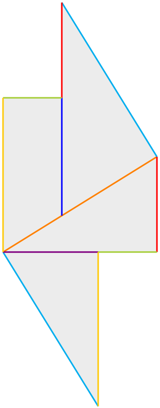
Unique ergodicity for periodic type

Consider the straight-line flow on a translation surface in direction θ. We can rotate the surface so that the flow is vertically upwards. Cut and paste to make the surface out of rectangles:



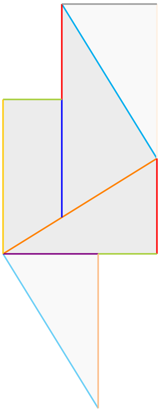
Unique ergodicity for periodic type

Consider the straight-line flow on a translation surface in direction θ. We can rotate the surface so that the flow is vertically upwards. Cut and paste to make the surface out of rectangles:



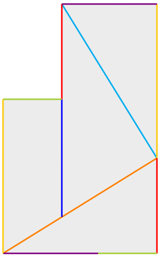
Unique ergodicity for periodic type

Consider the straight-line flow on a translation surface in direction θ. We can rotate the surface so that the flow is vertically upwards. Cut and paste to make the surface out of rectangles:



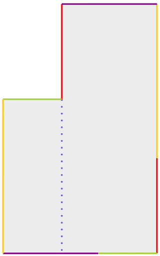
Unique ergodicity for periodic type

Consider the straight-line flow on a translation surface in direction θ. We can rotate the surface so that the flow is vertically upwards. Cut and paste to make the surface out of rectangles:



Unique ergodicity for periodic type

Consider the straight-line flow on a translation surface in direction θ. We can rotate the surface so that the flow is vertically upwards. Cut and paste to make the surface out of rectangles:

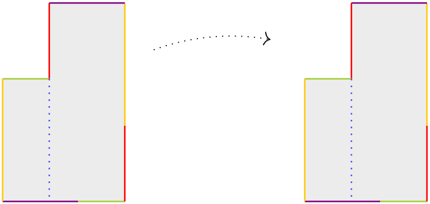


Unique ergodicity for periodic type

Now suppose the orbit of the surface under Teichmüller geodesic flow is periodic, i.e. after stretching the surface for some time t, we can cut and re-glue to get the original back:

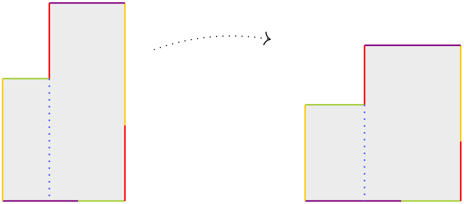
Unique ergodicity for periodic type

Now suppose the orbit of the surface under Teichmüller geodesic flow is periodic, i.e. after stretching the surface for some time t, we can cut and re-glue to get the original back:



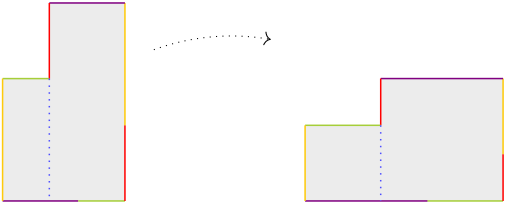
Unique ergodicity for periodic type

Now suppose the orbit of the surface under Teichmüller geodesic flow is periodic, i.e. after stretching the surface for some time t, we can cut and re-glue to get the original back:



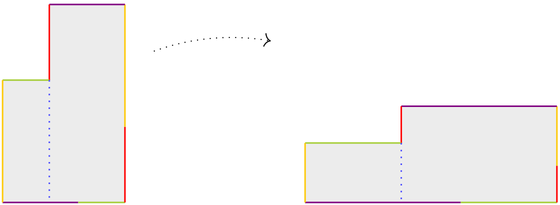
Unique ergodicity for periodic type

Now suppose the orbit of the surface under Teichmüller geodesic flow is periodic, i.e. after stretching the surface for some time t, we can cut and re-glue to get the original back:



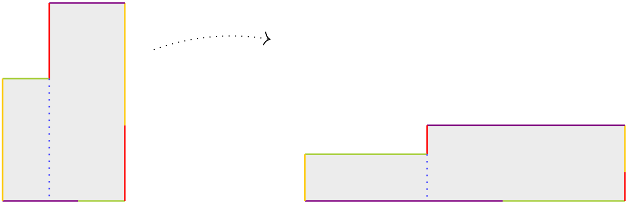
Unique ergodicity for periodic type

Now suppose the orbit of the surface under Teichmüller geodesic flow is periodic, i.e. after stretching the surface for some time t, we can cut and re-glue to get the original back:



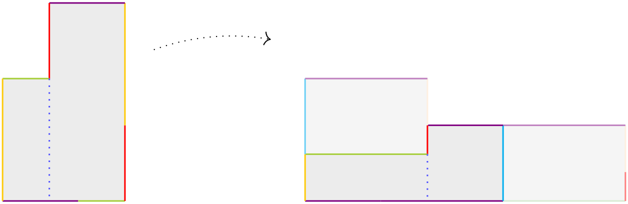
Unique ergodicity for periodic type

Now suppose the orbit of the surface under Teichmüller geodesic flow is periodic, i.e. after stretching the surface for some time t, we can cut and re-glue to get the original back:



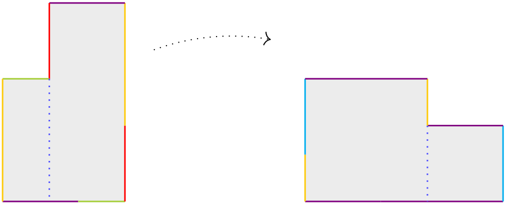
Unique ergodicity for periodic type

Now suppose the orbit of the surface under Teichmüller geodesic flow is periodic, i.e. after stretching the surface for some time t, we can cut and re-glue to get the original back:



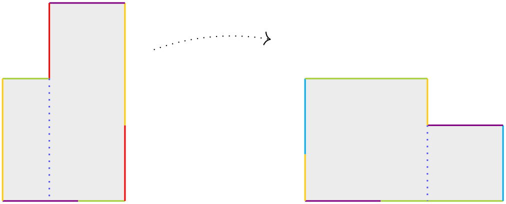
Unique ergodicity for periodic type

Now suppose the orbit of the surface under Teichmüller geodesic flow is periodic, i.e. after stretching the surface for some time t, we can cut and re-glue to get the original back:



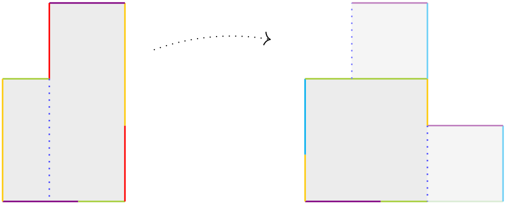
Unique ergodicity for periodic type

Now suppose the orbit of the surface under Teichmüller geodesic flow is periodic, i.e. after stretching the surface for some time t, we can cut and re-glue to get the original back:



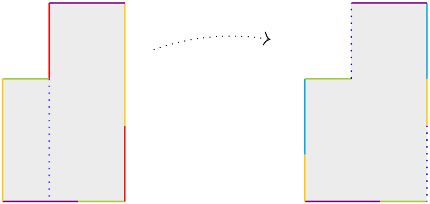
Unique ergodicity for periodic type

Now suppose the orbit of the surface under Teichmüller geodesic flow is periodic, i.e. after stretching the surface for some time t, we can cut and re-glue to get the original back:



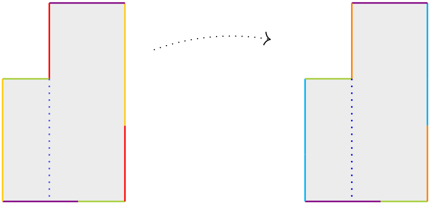
Unique ergodicity for periodic type

Now suppose the orbit of the surface under Teichmüller geodesic flow is periodic, i.e. after stretching the surface for some time t, we can cut and re-glue to get the original back:



Unique ergodicity for periodic type

Now suppose the orbit of the surface under Teichmüller geodesic flow is periodic, i.e. after stretching the surface for some time t, we can cut and re-glue to get the original back:

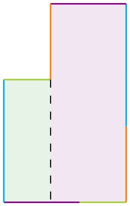


Unique ergodicity for periodic type

Going backwards, this allows us to divide the surface into thinner strips:

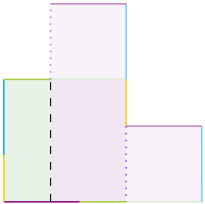
Unique ergodicity for periodic type

Going backwards, this allows us to divide the surface into thinner strips:



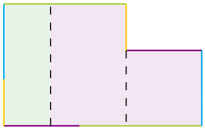
Unique ergodicity for periodic type

Going backwards, this allows us to divide the surface into thinner strips:



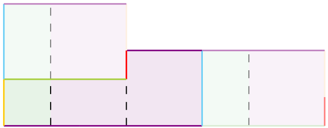
Unique ergodicity for periodic type

Going backwards, this allows us to divide the surface into thinner strips:



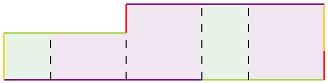
Unique ergodicity for periodic type

Going backwards, this allows us to divide the surface into thinner strips:



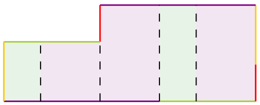
Unique ergodicity for periodic type

Going backwards, this allows us to divide the surface into thinner strips:



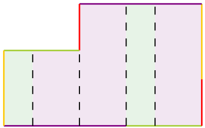
Unique ergodicity for periodic type

Going backwards, this allows us to divide the surface into thinner strips:



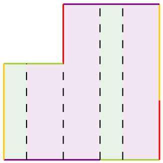
Unique ergodicity for periodic type

Going backwards, this allows us to divide the surface into thinner strips:



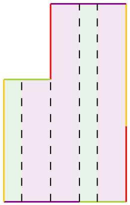
Unique ergodicity for periodic type

Going backwards, this allows us to divide the surface into thinner strips:



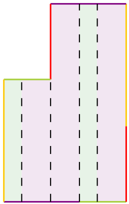
Unique ergodicity for periodic type

Going backwards, this allows us to divide the surface into thinner strips:



Unique ergodicity for periodic type

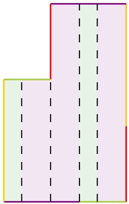
Going backwards, this allows us to divide the surface into thinner strips:



If μ is an invariant measure, then the μ-width of each strip is constant as one moves vertically along the strip.

Unique ergodicity for periodic type

Going backwards, this allows us to divide the surface into thinner strips:



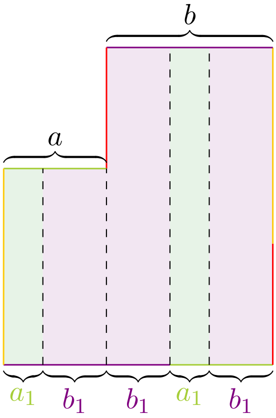
If μ is an invariant measure, then the μ-width of each strip is constant as one moves vertically along the strip.
The green strips came from one strip, so have the same μ-width a_1.
Similarly the purple strips all have the same μ-width b_1.

Unique ergodicity for periodic type

Let a, b be the widths of the initial rectangles:

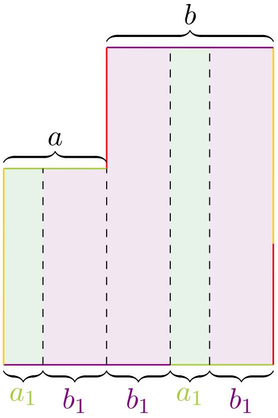
Unique ergodicity for periodic type

Let a, b be the widths of the initial rectangles:



Unique ergodicity for periodic type

Let a, b be the widths of the initial rectangles:



Conclude:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}.$$

Unique ergodicity for periodic type

Key step: since surface is periodic under Teichmüller geodesic flow, we can do the same thing again!

Unique ergodicity for periodic type

Key step: since surface is periodic under Teichmüller geodesic flow, we can do the same thing again!

Two steps of flow and cutting/re-gluing gives thinner strips of widths a_2, b_2, and

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^2 \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

Unique ergodicity for periodic type

Key step: since surface is periodic under Teichmüller geodesic flow, we can do the same thing again!

n steps of flow and cutting/re-gluing gives thinner strips of widths a_n, b_n, and

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

Unique ergodicity for periodic type

Key step: since surface is periodic under Teichmüller geodesic flow, we can do the same thing again!

n steps of flow and cutting/re-gluing gives thinner strips of widths a_n, b_n, and

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

But widths have to be non-negative!

So for any n, $\begin{pmatrix} a \\ b \end{pmatrix} \in \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \cdot \mathbb{R}^2_+$.

Unique ergodicity for periodic type

Key step: since surface is periodic under Teichmüller geodesic flow, we can do the same thing again!

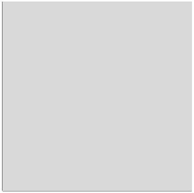
n steps of flow and cutting/re-gluing gives thinner strips of widths a_n, b_n, and

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

But widths have to be non-negative!

So for any n, $\begin{pmatrix} a \\ b \end{pmatrix} \in \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \cdot \mathbb{R}^2_+$.

$\mathbb{R}^2_+$



Unique ergodicity for periodic type

Key step: since surface is periodic under Teichmüller geodesic flow, we can do the same thing again!

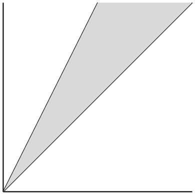
n steps of flow and cutting/re-gluing gives thinner strips of widths a_n, b_n, and

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

But widths have to be non-negative!

So for any n, $\begin{pmatrix} a \\ b \end{pmatrix} \in \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \cdot \mathbb{R}^2_+$.

$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \cdot \mathbb{R}^2_+$



Unique ergodicity for periodic type

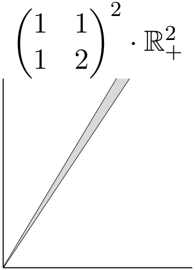
Key step: since surface is periodic under Teichmüller geodesic flow, we can do the same thing again!

n steps of flow and cutting/re-gluing gives thinner strips of widths a_n, b_n, and

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

But widths have to be non-negative!

So for any n, $\begin{pmatrix} a \\ b \end{pmatrix} \in \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \cdot \mathbb{R}^2_+$.



Unique ergodicity for periodic type

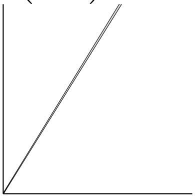
Key step: since surface is periodic under Teichmüller geodesic flow, we can do the same thing again!

n steps of flow and cutting/re-gluing gives thinner strips of widths a_n, b_n, and

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \begin{pmatrix} a_n \\ b_n \end{pmatrix}.$$

But widths have to be non-negative!

So for any n, $\begin{pmatrix} a \\ b \end{pmatrix} \in \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \cdot \mathbb{R}^2_+$.



Unique ergodicity for periodic type

Since

$$\bigcap_{n\geq 1} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \cdot \mathbb{R}^2_+$$

is a line, there is a unique possible ratio for $\frac{b}{a}$.

Unique ergodicity for periodic type

Since

$$\bigcap_{n\geq 1} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \cdot \mathbb{R}^2_+$$

is a line, there is a unique possible ratio for $\frac{b}{a}$.

From a, b we can determine a_n, b_n by

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-n} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Unique ergodicity for periodic type

Since

$$\bigcap_{n\geq 1} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \cdot \mathbb{R}^2_+$$

is a line, there is a unique possible ratio for $\frac{b}{a}$.

From a, b we can determine a_n, b_n by

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-n} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Can approximate any strip using a_n, b_n with large n, hence can determine μ.

Unique ergodicity for periodic type

Since

$$\bigcap_{n\geq 1} \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^n \cdot \mathbb{R}^2_+$$

is a line, there is a unique possible ratio for $\frac{b}{a}$.

From a, b we can determine a_n, b_n by

$$\begin{pmatrix} a_n \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-n} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Can approximate any strip using a_n, b_n with large n, hence can determine μ.

Thus (up to scaling) there is a unique invariant measure for the vertical flow, so it is uniquely ergodic.

Unique ergodicity for periodic type

In general need to use:

Theorem (Perron-Frobenius theorem)

The top eigenvalue of any positive matrix is simple.

Unique ergodicity for periodic type

In general need to use:

Theorem (Perron-Frobenius theorem)

The top eigenvalue of any positive matrix is simple.

To prove:

Theorem

If S is a translation surface that is periodic under Teichmüller geodesic flow, then the vertical straight-line flow on S is uniquely ergodic. The unique invariant measure is Lebesgue.

Stronger results

Theorem (Masur's criterion, Howard Masur '82)

If the surface S returns to some compact set infinitely many times under Teichmüller geodesic flow, then the vertical flow on S is uniquely ergodic.

Stronger results

Theorem (Masur's criterion, Howard Masur '82)

If the surface S returns to some compact set infinitely many times under Teichmüller geodesic flow, then the vertical flow on S is uniquely ergodic.

Theorem (Kerckhoff, Masur, Smillie '86)

For any translation surface S, for almost any direction θ, the straight-line flow in direction θ is uniquely ergodic.

Thanks for your attention!