Counting: Choosing

I'm painting, and want to pick two colours to use out of the five (red, green, blue, yellow and purple) that I have (the order doesn't matter!) How many ways are there to do this?

You can check the number of ways is 10, so we say '5 choose 2' is 10, and write $\binom{5}{2} = 10$. (Number of colours I have goes at the top, the number I need to pick at the bottom). These are called 'binomial coeficients' but we'll just say 'choosing numbers'.

We say $\binom{1}{0} = \binom{2}{0} = ... = \binom{5}{0}... = 1$ (whether I have 5 colours or however many, there is only one way to choose no colours - by choosing nothing), and similarly $\binom{1}{1} = \binom{2}{2} = ... = \binom{5}{5}... = 1$ (there is only one way to pick all of the colours you have).

Also we can see that $\binom{5}{1} = \binom{5}{4}, \binom{5}{2} = \binom{5}{3}$, why is that? Because picking two colours out of five to use for my painting is the same thing as picking the three colours to not use!

What happens if we write these numbers in a table, like this:

$$\begin{array}{llllll}
\binom{0}{0} \\
\binom{1}{0} & \binom{1}{1} \\
\binom{2}{0} & \binom{2}{1} & \binom{2}{2} \\
\binom{3}{0} & \binom{3}{1} & \binom{3}{2} & \binom{3}{3} \\
\binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \\
\binom{5}{0} & \binom{5}{1} & \binom{5}{2} & \binom{5}{3} & \binom{5}{4} & \binom{5}{5}
\end{array}
\qquad
\begin{array}{llllll}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1
\end{array}$$

This is just Pascal's triangle with the rows shifted to the left! Why is that? Let's think what happens when we add these choosing munbers.

When I'm choosing three colours to use out of 5, I can either decide to pick red, or decide to not pick red.
If I pick red, I need 2 more colours, out of the four which are left, so that's $\binom{4}{2}$ choices if I use red.
But if I decide to not use red, then I still have only four colours left to choose from, but need to pick 3 out of them. So I have $\binom{4}{3}$ choices if I don't use red.

So we know that $\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$, and similarly for example $\binom{7}{2} = \binom{6}{1} + \binom{6}{2}$ and so on.

So if we write the choosing numbers in a triangle:

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

Then the numbers next to each other add up to give the one below them, and because the ends of each row are 1s, we get Pascal's triangle.

There's also another way to see this: let's count how many paths there are from the top of the triangle to the first 10 (it's the second number in the fifth row if we start counting from 0), where we go down a row each step:

$$
\begin{array}{lcccccccccccl}
\text{0th row} & & & & & & \color{red}{1} & & & & & & \\
\text{1st row} & & & & & 1 & & \color{red}{1} & & & & & \\
\text{2nd row} & & & & 1 & & \color{red}{2} & & 1 & & & & \\
\text{3rd row} & & & 1 & & \color{red}{3} & & 3 & & 1 & & & \\
\text{4th row} & & 1 & & \color{red}{4} & & 6 & & 4 & & 1 & & \\
\text{5th row} & 1 & & 5 & & \color{red}{10} & & 10 & & 5 & & 1 & \\
 & \text{0th} & & \text{1st} & & \text{2nd} & & \text{3rd} & & \text{4th} & & \text{5th} & \text{number}
\end{array}
$$

We can see that we need 5 steps, and each time we go either left or right, so for example the path in the picture is RLLLR. To end up at the second number we need to go right twice. But we can choose any two out of the 5 steps we have to go right (like here we chose steps 1 and 5), and so there are $\binom{5}{2}$ paths.