# Groups Acting on Trees

Yuriy Tumarkin

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## 0 Introduction

This essay is an exposition of a few of the main results concerning the actions of groups on trees, following Serre's book [4]. In chapter 1 we consider free actions and prove that only free groups act freely on trees. From this one can directly deduce Schreier's theorem, that any subgroup of a free group is free.

The second part of the essay (chapters 2 and 3) is a presentation of Bass-Serre theory, which links the structure of a group with its actions on trees. Namely, an action of a group $G$ on a tree $X$ defines a decomposition of $G$ called a graph of groups, $(\mathcal{G}, Y)$. Conversely, from a graph of groups $(\mathcal{G}, Y)$ we can define a group $\pi$ (called the fundamental group) and a tree $\tilde{X}$ on which $\pi$ acts. The main result is the statement that these two constructions are inverse to each other, i.e. that a decomposition into a graph of groups uniquely corresponds to an action on a tree. This correspondence then gives a structure theorem for all groups acting on trees.

## 1 Free actions

### 1.1 Graphs

Before considering group actions on graphs we need to define what we mean by a graph. We will use Serre's formalism, where a graph is a 1-complex (so it can have loops and multiple edges), and edges are directed and always come in pairs $\{y, \bar{y}\}$, where $\bar{y}$ is the reverse of $y$.

**Definition 1.** A **graph** $\Gamma$ consists of:

- Two sets, vertices $\mathrm{V}(\Gamma)$ and edges $\mathrm{E}(\Gamma)$.
- The endpoint maps $\mathrm{o}, \mathrm{t} : \mathrm{E}(\Gamma) \to \mathrm{V}(\Gamma)$ $\big($o stands for 'origin' and t for 'terminus', so edge $y$ goes from $\mathrm{o}(y)$ to $\mathrm{t}(y)\big)$.
- An involution $^{-} : \mathrm{E}(\Gamma) \to \mathrm{E}(\Gamma)$, satisfying $\forall y \in \mathrm{E}(\Gamma)\ \bar{y} \neq y,\ \mathrm{o}(\bar{y}) = \mathrm{t}(y),\ \mathrm{t}(\bar{y}) = \mathrm{o}(y)$.

An **orientation** of a graph $\Gamma$ is a choice $E_+ \subset \mathrm{E}(\Gamma)$ such that $\forall y \in \mathrm{E}(\Gamma)$ exactly one of $y, \bar{y}$ lies in $E_+$.

A **path** of length $n$ in $\Gamma$ is a sequence of edges $(y_1, ..., y_n)$, where for $1 \leq i \leq n-1,\ \mathrm{t}(y_i) = \mathrm{o}(y_{i+1})$. We say $c$ has a **backtracking** if for some $i,\ y_{i+1} = \bar{y}_i$.

For a path $c = (y_1, ..., y_n)$, define $\mathrm{o}(c) = \mathrm{o}(y_1)$ and $\mathrm{t}(c) = \mathrm{t}(y_n)$. A **cycle** is a path $c$ without backtracking, which has $\mathrm{o}(c) = \mathrm{t}(c)$.

A **tree** is a non-empty connected graph which does not contain cycles. (When saying that a graph is connected, we mean that its natural realisation is connected in the topological sense. Alternatively we can use path-connectedness: we can say that $\Gamma$ is connected if for any vertices $P, Q \in \mathrm{V}(\Gamma)$ there exists a path $c$ with $\mathrm{o}(c) = P,\ \mathrm{t}(c) = Q$.)

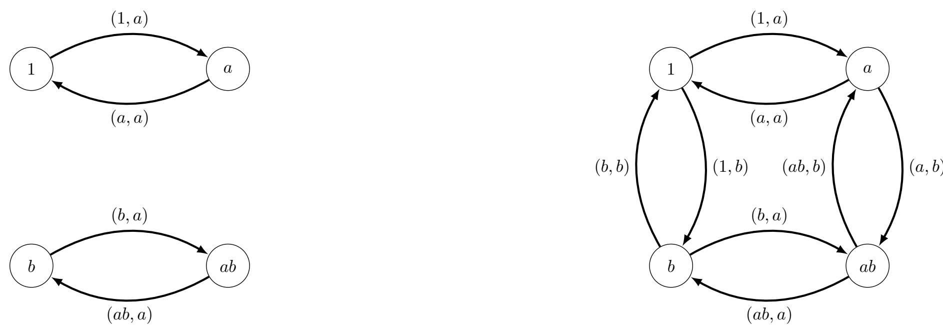
A **maximal tree** of a graph is a subtree that is maximal under inclusion. A maximal tree of a graph $X$ contains all vertices of $X$, else it could be extended.

Consider a group $G$ and a set $S \subset G$. We can define an oriented graph $X = \Gamma(G, S)$ with $\mathrm{V}(X) = G$, $E_+ = G \times S$, where for $y = (g, s) \in E_+$, $\mathrm{o}(y) = g$, $\mathrm{t}(y) = gs$.

For $y = (g, s) \in E_+$ denote $\bar{y}$ by $(\widehat{gs, s^{-1}})$ (the hat is to avoid potential confusion with the edge $(gs, s^{-1})$, which lies in $E_+$ if also $s^{-1} \in S$).

Note that $G$ acts freely on $\Gamma(G, S)$ via $h : g \mapsto hg$, $h : (g, s) \mapsto (hg, s)$.

*Examples:*




(a) $S = \{a\}$. (b) $S = \{a, b\}$.

Figure 1: The oriented graphs for $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle a, b \mid a^2 = b^2 = abab = 1\rangle$ with different sets $S$ (only the edges in $E_+$ are shown).

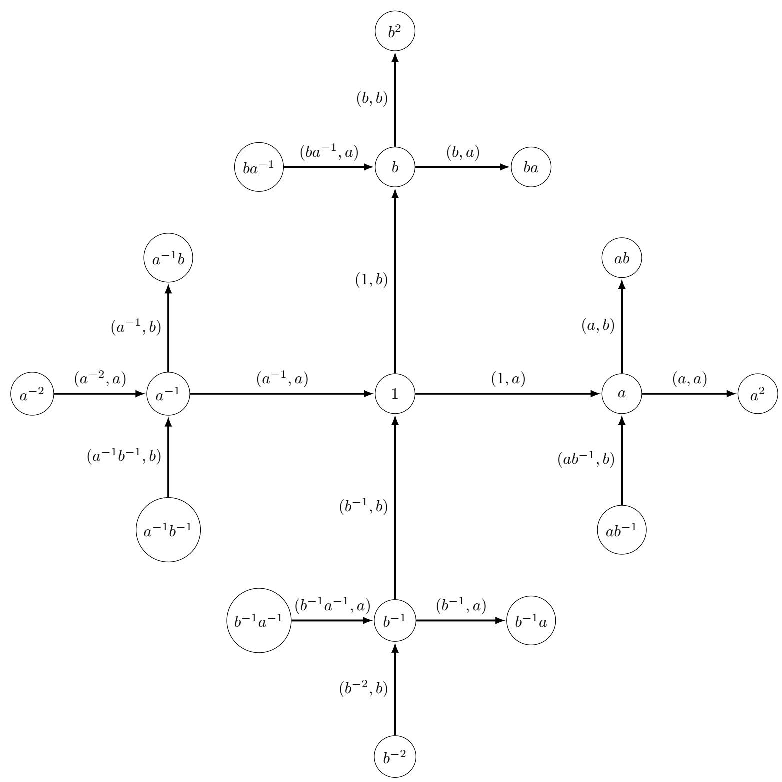



Figure 2: A part of the graph for $G = \mathbb{F}_2 = \langle a, b \rangle$ with $S = \{a, b\}$.

**Proposition 1.** Defined as above,

(i.) Cycles in $X$ correspond to relations on elements of $S$.

(ii.) $X$ is connected $\iff$ $S$ generates $G$.

From these it immediately follows that $X$ is a tree if and only if $S$ generates $G$ freely.

*Proof.* Consider a path without backtracking $c = (y_1, ..., y_n)$ in $X$ with $y_i$ either $(g_i, s_i)$ or $\widehat{(g_i, s_i)}$ for some $g_i \in G,\ s_i \in S \cup S^{-1}$. Then $\mathrm{o}(c) = g_1$ and by induction $g_i = g_1 s_1 s_2 ... s_{i-1}$, so $\mathrm{t}(c) = g_1 s_1 ... s_n$.

(i.) $c$ is a cycle if and only if $s_1 ... s_n = 1$, so cycles correspond to relations on $S$.

(ii.) $X$ is connected if and only if for any $g \in G$ there is a path $c_g$ in $X$ with $\mathrm{o}(c_g) = 1,\ \mathrm{t}(c_g) = g$. Such a path exists for $g$ if and only if $g = s_1 s_2 ... s_n$ for some $s_i \in S \cup S^{-1}$.
Indeed, if $c_g$ exists, then it must have the form of $c$ above, and hence $g = s_1 s_2 ... s_n$ for some $s_i \in S \cup S^{-1}$.
Conversely, if $g = s_1 s_2 ... s_n$, then let $y_i = (s_1 ... s_{i-1}, s_i)$ if $s_i \in S$, or $y_i = (\widehat{s_1 ... s_{i-1}, s_i})$ if $s_i \in S^{-1}$. Then $c_g = (y_1, ..., y_n)$.
Hence $c_g$ exists for all $g \in G$ if and only $S$ generates $G$. $\square$

## 1.2 Trees of representatives

If a group $G$ acts on a graph $X$, a very useful object to consider is the quotient $G \backslash X$. $Y = G \backslash X$ has vertices $\mathrm{V}(Y) = G \backslash \mathrm{V}(X)$ and edges $\mathrm{E}(Y) = G \backslash \mathrm{E}(Y)$, and it also inherits a map $^- : \mathrm{E}(Y) \to \mathrm{E}(Y)$, with $\overline{G \cdot y} = G \cdot \bar{y}$. However it is not necessarily true that $\bar{z} \neq z$ for $z \in \mathrm{E}(Y)$. Hence $Y$ is not necessarily a graph. This is an unnecessary complication, so we only consider actions where $G \cdot y \neq G \cdot \bar{y}$ for all $y \in \mathrm{E}(X)$.

**Definition 2.** We say group $G$ acts on graph $X$ **without inversion** if for all edges $y \in \mathrm{E}(X)$ and elements $g \in G$, $gy \neq \bar{y}$.

From now on, we will only consider actions without inversion. This is not restrictive, since for any action of $G$ on $X$, we can barycentrically subdivide $X$ to obtain an action without inversion (see Figure 3). It is easy to check that if $G$ acts without inversion on $X$ then $G \backslash X$ is indeed a graph.

If $G$ acts on $X$ without inversion, then there is an orientation $E_+$ of $X$ that is preserved under $G$. Indeed, in every orbit, we pick the orientation of one edge as we wish, and then the orientation is uniquely induced for all the other edges.

*Example:*

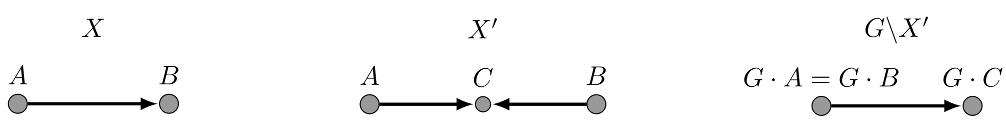



Figure 3: The group $G = \mathbb{Z}/2\mathbb{Z}$ acts on $X$ in the obvious way, swapping $A$ and $B$. Taking the barycentric subdivision $X'$ of $X$ we obtain an action without inversion, and hence there is a quotient graph $G \backslash X'$. The diagram for $X'$ shows an example of an orientation that is preserved by $G$, and the orientation on $G \backslash X'$ is inherited from $X'$. (Again we only show the positive edges).

**Proposition 2.** Suppose group $G$ acts on a connected graph $X$. Then every subtree $T$ of $Y = G \backslash X$ lifts to a subtree of X.

*Proof.* Let $q : X \to Y$ be the quotient map.
Consider the set $\Omega = \{\text{trees } \widehat{T} \text{ in } X \text{ such that } q(\widehat{T}) \subset T \text{ and } q \text{ is injective on } \widehat{T}\}$. Then $\Omega$ is non-empty since it contains single vertices of $q^{-1}(T)$. Hence we can take $\widehat{T_0}$, a maximal element of $\Omega$ under inclusion. Let $T_0 = q(\widehat{T_0})$.

Suppose $T_0 \neq T$. Then some edges of $T$ are missing in $T_0$, and we can pick one which starts in $T_0$. Thus there is an edge $y \in \mathrm{E}(T) \setminus \mathrm{E}(T_0)$, with $\mathrm{o}(y) \in T_0$. Let $P = \mathrm{t}(y)$, then $P$ is not in $T_0$ or else $T$ would not be a tree.

Let $\widehat{y}$ be any lift of $y$ to $X$. As $q(\mathrm{o}(\widehat{y})) = \mathrm{o}(y) \in T_0$, there is a $g \in G$ such that $g\mathrm{o}(\widehat{y}) \in \widehat{T_0}$. Then $g\widehat{y}$ is a lift of $y$ starting in $\widehat{T_0}$. So we can pick a lift $\widehat{y}$ with $\mathrm{o}(\widehat{y}) \in \widehat{T_0}$.

Now let $\widehat{P} = \mathrm{t}(\widehat{y})$ and add $\widehat{P}$ and $\widehat{y}$ to $\widehat{T_0}$ to get the graph $\widehat{T_1} = \widehat{T_0} + \{\widehat{y}, \bar{\widehat{y}}\} + \widehat{P}$.
As $\widehat{P} \notin \widehat{T_0}$ (since $P \notin T_0$), $\widehat{T_1}$ is also a tree. $q(\widehat{T_1}) \subset T$ and $q$ is still injective on $\widehat{T_1}$ (as $y, P \notin T_0$). Hence we get a contradiction.

Thus if $\widehat{T_0}$ is a maximal element of $\Omega$, $q(\widehat{T_0}) = T$, so a required lift of $T$ exists. $\square$

**Definition 3.** If $G$ acts on $X$, a **tree of representatives** of $X \bmod G$ is a lift $T$ of a maximal subtree $T'$ of $G\backslash X$.

Suppose $T$ is a tree of representatives of $X \bmod G$. Then since the projection of $T$ in $G\backslash X$ is maximal, it contains all vertices of $G\backslash X$. Hence $G \cdot T$ contains all vertices of $X$.

## 1.3 Contracting

The final ingredient we need for the main theorem of this section is the idea of contracting a subtree of a graph. If $T$ is a subtree of $X$, we can replace the entire tree $T$ by a single vertex $(T)$. We remove the edges in $T$, and for the edges with one endpoint in $T$, this endpoint is now $(T)$. Further we can simultaneously contract multiple disjoint trees.

Formally,

**Definition 4.** Let $X$ a graph, and consider a subgraph $Y$ that is a union of disjoint trees, $Y = \bigcup_{i \in I} T_i$. Then the **contraction** $X/Y$ is the following graph:

- $\mathrm{V}(X/Y) = \mathrm{V}(X)/R$, where $R$ is the relation with the set of equivalence classes $\{T_i : i \in I\} \cup \{v \in \mathrm{V}(X) \setminus \mathrm{V}(Y)\}$.
- $\mathrm{E}(X/Y) = \mathrm{E}(X) \setminus \mathrm{E}(Y)$.
- The reverse map $^- : \mathrm{E}(X/Y) \to \mathrm{E}(X/Y)$ is inherited from $X$, and the maps $\mathrm{o}, \mathrm{t} : \mathrm{E}(X/Y) \to \mathrm{V}(X/Y)$ are induced by the quotients.

*Remark.* Note that this corresponds to the topological notion of quotienting by a subspace, in fact the realisation of $X/Y$ is the quotient $\mathrm{real}(X/Y) = \mathrm{real}(X)/\mathrm{real}(Y)$.

Now we use a fact from algebraic topology (see for example [2], p. 15 for reference).

**Proposition 3.** If $X$ is a CW-complex and $A$ is a contractible subcomplex, then the quotient $X \to X/A$ is a homotopy equivalence.

The realisation of a graph is naturally a CW-complex, and the realisation of a subgraph is a subcomplex. Since a graph is a tree if and only if its realisation is contractible, this implies that if $X$ is a tree and $Y$ is a union of disjoint subtrees, then the contraction $X/Y$ is also a tree.

## 1.4 Groups acting freely on graphs

Now we can prove the following theorem:

**Theorem 1.** Suppose $G$ acts freely on a tree $X$. Let $T$ be a tree of representatives of $X$ mod $G$, and let $E_+$ be an orientation on $X$ preserved by $G$. Then if

$$S = \{1 \neq g \in G : \exists y \in E_+ \text{ with } \mathrm{o}(y) \in T, \mathrm{t}(y) \in gT\},$$

then $S$ freely generates $G$.

*Proof.* Consider $a \neq b \in G, P \in \mathrm{V}(T)$. As $T$ is a tree of representatives, $G{\cdot}P \cap T = P$. Hence $G{\cdot}P \cap aT = aP$, and as $G$ acts freely, $bP \neq aP$, so $bP \notin aT$. Hence $aT \ \cap\ bT = \emptyset$. Thus the translates $gT$ are all pairwise disjoint, and there is a bijection $g \mapsto gT$.

We can contract the trees $gT$ to make $X' = X/(G \cdot T)$, with vertices $\{(gT) : g \in G\}$. We know that $X'$ is a tree, and we will show that it is isomorphic to $\Gamma(G,S)$, which will give us the required result by Prop. 1.

We already have a bijection $\alpha : \mathrm{V}(X') \to \mathrm{V}(\Gamma(G,S)) = G$ given by $(gT) \mapsto g$. We shall extend it to a map of graphs.
The edges of $X'$ are $\mathrm{E}(X) \setminus \mathrm{E}(G \cdot T)$. Consider the inherited orientation on $X'$, $E'_+ = E_+ \cap \mathrm{E}(X')$. As the positively oriented edges of $\Gamma(G,S)$ are $G \times S$, it suffices to extend $\alpha$ to $\alpha : E'_+ \to G \times S$.

Now, for $y \in E'_+$, we have $\mathrm{o}(y) = (gT), \mathrm{t}(y) = (hT)$ for some $g, h \in G$. Then in $X$ the edge $y$ starts in $gT$ and ends in $hT$, so $g^{-1}y$ starts in $T$ and ends in $g^{-1}hT$. Hence $g^{-1}h \in S$. Thus we can define $\alpha(y) = (g, g^{-1}h)$.

Then $\mathrm{o}(\alpha(y)) = g = \alpha(\mathrm{o}(y))$, and $\mathrm{t}(\alpha(y)) = h = \alpha(\mathrm{t}(y))$, so $\alpha$ is a valid map of graphs.

Further, $\alpha$ is injective as for any $g, h \in G$ there can be at most one edge in $X$ starting in $gT$ and ending in $hT$ since $X$ is a tree.

And $\alpha$ is surjective as for any $g \in G, s \in S$, there is an edge $y \in E'_+$ starting in $T$ and ending in $sT$, so $gy$ starts in $gT$ and ends in $gsT$. Then $\alpha(gy) = (g, s)$.

Hence $\alpha$ is an isomorphism, which shows that $\Gamma(G,S)$ is a tree, and hence we are done. $\square$

**Corollary.** Any subgroup of a free group is free.

*Proof.* If $G$ is free, then it acts freely on the tree $\Gamma(G,S)$, where $S$ is a basis for $G$. Then any subgroup $H \leqslant G$ also acts on $\Gamma(G,S)$, and hence $H$ must be free. $\square$

*Example:*

Let $G$ be the free group on two elements $\mathbb{F}_2 = \langle a, b\rangle$, let $n \geqslant 2$, and consider the homomorphism
$\phi : G \to \mathbb{Z}/n\mathbb{Z} = \langle c\rangle$ given by $a \mapsto c,\ b \mapsto 1$.

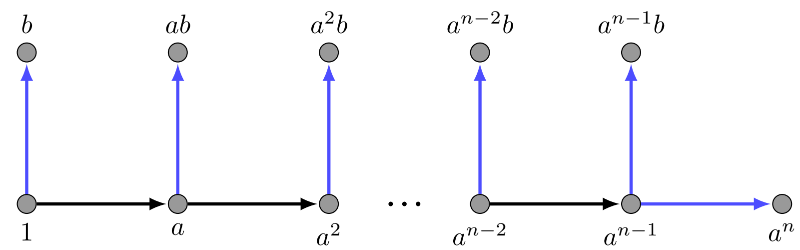
Let $H = \ker \phi$.

Then if $X = \Gamma(G, \{a, b\})$, then the vertices of $H\backslash X$ are the cosets of $H$ in $G$, which are $H, aH, ..., a^{n-1}H$. Hence the subtree

$$1 \longrightarrow a \longrightarrow a^2 \quad \cdots \quad a^{n-2} \longrightarrow a^{n-1}$$

is a tree of representatives of $X$ mod $H$. Call it $T$.

Consider the positively oriented edges coming out of it (shown in blue):




Now, $a^kb = a^kba^{-k}a^k$, so the vertex $a^kb$ belongs to $a^kba^{-k}T$. And $a^n \in a^nT$.
Hence $S = \{a^kba^{-k} : 0 \leqslant k \leqslant n-1\} \cup \{a^n\}$, and $|S| = n+1$.

Thus in particular we have shown that $\mathbb{F}_2$ contains a subgroup isomorphic to $\mathbb{F}_m$ for any $m \geqslant 3$.

# 2 Graphs of groups

If a group $G$ acts on a graph $X$, we can consider the stabilisers $G_P$ and $G_y$ for vertices $P$, edges $y$. Clearly $G_y = G_{\mathrm{o}(y)} \cap G_{\mathrm{t}(y)}$, so we have injective homomorphisms $G_y \hookrightarrow G_{\mathrm{o}(y)}$ and $G_y \hookrightarrow G_{\mathrm{t}(y)}$. Also note that $G_y = G_{\bar{y}}$.

The structure consisting of the underlying graph, the vertex and edge stabilisers, and the inclusions $G_y \hookrightarrow G_{\mathrm{o}(y)}$ and $G_y \hookrightarrow G_{\mathrm{t}(y)}$ is an example of a graph of groups, which we will define formally shortly.

Bass-Serre theory explores the relationship between graphs of groups and actions on graphs, and we will see in chapter 3 that there are natural constructions that give a correspondence between the two. In this chapter we define the main ingredients of these constructions (2.1), look at some motivating examples (2.2), and prove a technical lemma that will be useful later (2.3).

## 2.1 Definitions

**Definition 5.** If $Y$ is a connected non-empty graph, a **graph of groups** $(\mathcal{G}, Y)$ is a collection of groups $G_P$ for $P \in \mathrm{V}(Y)$, $G_y$ for $y \in \mathrm{E}(Y)$ with $G_y = G_{\bar{y}}$, and injective homomorphisms $G_y \hookrightarrow G_{\mathrm{o}(y)}$ and $G_y \hookrightarrow G_{\mathrm{t}(y)}$.

We will usually denote the homomorphism $G_y \hookrightarrow G_{\mathrm{t}(y)}$ by $a \mapsto a^y$ (and hence $G_y \hookrightarrow G_{\mathrm{o}(y)}$ by $a \mapsto a^{\bar{y}}$).

Now since we can treat a graph of groups as the underlying graph with additional structure, it is natural to consider paths in it. Since the additional structure consists of the groups, we wish to preserve this information, so our 'paths' will consist of an edge path, but with an element of $G_P$ chosen at every vertex $P$ along the path. Thus 'paths' will be alternating words of edges $y$ and elements of relevant vertex groups.

Hence we will define a group $F(\mathcal{G}, Y)$ where words of a certain format will correspond to these 'paths'.

Formally:

**Definition 6.** Given a graph of groups $(\mathcal{G}, Y)$, the group $\boldsymbol{F(\mathcal{G}, Y)}$ is generated by the vertex groups $G_P$ for $P \in \mathrm{V}(Y)$ and the edges $y$ for $y \in \mathrm{E}(Y)$, with the relations:

$$ya^y\bar{y} = a^{\bar{y}} \text{ for all } y \in \mathrm{E}(Y),\ a \in G_y.$$

(Note in particular for $a = 1$ we get $y^{-1} = \bar{y}$.)

This definition is designed to generalise both free products with amalgamations and HNN extensions (see 2.2.3), hence these relations. In 2.2.1 we will see why they make sense from a topological point of view.

**Definition 7.** The **fundamental group of a graph of groups** $(\mathcal{G}, Y)$ relative to a maximal subtree $T$ of $Y$ is the quotient $\pi_1(\mathcal{G}, Y, T) = F(\mathcal{G}, Y)/N$, where $N$ is the normal closure of the free subgroup with basis $\{y : y \in \mathrm{E}(T)\}$.

Thus $\pi_1(\mathcal{G}, Y, T)$ is generated by $G_P$ for $P \in \mathrm{V}(Y)$ and elements $g_y$ for $y \in \mathrm{E}(Y)$ with $g_{\bar{y}} = g_y^{-1}$, $g_y a^y g_y^{-1} = a^{\bar{y}}$ for all $y \in \mathrm{E}(Y)$, $a \in G_y$, and in addition $g_y = 1$ for $y \in \mathrm{E}(T)$.
(Here $g_y$ is the image of $y$ under the quotient map).

$\pi_1(\mathcal{G}, Y, T)$ is the object we will deal with a lot later, but to see why it is called a 'fundamental group' we can define another object, the group of loops based at a vertex, and we will show that it is isomorphic to $\pi_1(\mathcal{G}, Y, T)$.

**Definition 8.** For a path $c$ in $Y$ of length $n$ with edges $y_1, ..., y_n$, vertices $P_i = \mathrm{o}(y_{i+1}) = \mathrm{t}(y_i)$, a **word of type *c*** in $(\mathcal{G}, Y)$ is $(c, \mu)$, where $\mu = (r_0, ..., r_n)$ with $r_i \in G_{P_i}$.

The **associated element** of $(c, \mu)$ is $|c, \mu| = r_0 y_1 ... y_n r_n \in F(\mathcal{G}, Y)$.

Then for $P_0 \in \mathrm{V}(Y)$ we define the **fundamental group of $(\mathcal{G}, Y)$ based at $P_0$** as

$$\pi_1(\mathcal{G}, Y, P_0) = \{|c, \mu| \in F(\mathcal{G}, Y) : \mathrm{o}(c) = \mathrm{t}(c) = P_0\}.$$

(Note that if $\mathcal{G}$ is trivial, i.e. every group is just $\{1\}$, then $\pi_1(\mathcal{G}, Y, P_0) = \pi_1(Y, P_0)$, where the latter is the fundamental group in the canonical topological sense.)

**Proposition 4.** There is an isomorphism between $\pi_1(\mathcal{G}, Y, P_0)$ and $\pi_1(\mathcal{G}, Y, T)$.

*Proof.* We will not give a full proof, but the idea is to map a vertex or edge to a path based at $P_0$ in the obvious way, by first going in $T$ from $P_0$ to the vertex/edge and then coming back.

For any vertex $P$ of $Y$, there is a unique path without backtracking from $P_0$ to $P$ in $T$. If the path is $(y_1, ..., y_n)$, let $\gamma_P = y_1 ... y_n \in F(\mathcal{G}, Y)$.

Then the homomorphism $f : \pi_1(\mathcal{G}, Y, T) \to \pi_1(\mathcal{G}, Y, P_0)$ defined by

$$\begin{aligned} x &\mapsto \gamma_P x \gamma_P^{-1} && \text{for } x \in G_P \\ g_y &\mapsto \gamma_{\mathrm{o}(y)} y \gamma_{\mathrm{t}(y)}^{-1} && \text{for } y \in \mathrm{E}(Y) \setminus \mathrm{E}(T) \end{aligned}$$

is well-defined and inverse to the restriction to $\pi_1(\mathcal{G}, Y, P_0)$ of the projection $F(\mathcal{G}, Y) \to \pi_1(\mathcal{G}, Y, T)$. $\square$

**Corollary.** Up to isomorphism, $\pi_1(\mathcal{G}, Y, P_0)$ is independent of the choice of vertex $P_0$, and similarly up to isomorphism $\pi_1(\mathcal{G}, Y, T)$ is independent of the choice of maximal tree $T$.

Picture for an edge:
(we abuse notation here, showing the paths in the graph corresponding to the elements of $\pi_1(\mathcal{G}, Y, T)$ and $\pi_1(\mathcal{G}, Y, P_0)$, in fact $f$ is a homomorphism of groups, not a map of graphs)

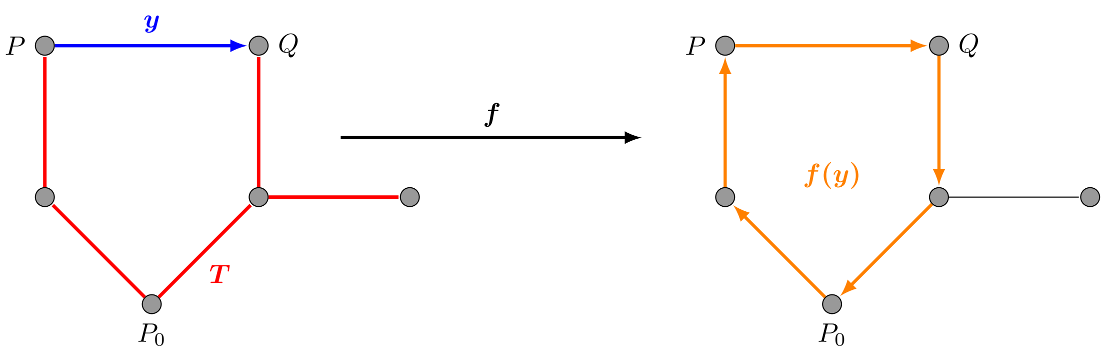


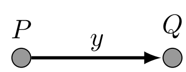

Figure 4: For an edge $y$ and a choice of maximal tree $T$ we show the path corresponding to $f(y)$.

## 2.2 Important examples of the fundamental group of graphs

Here we discuss the structure $\pi_1(\mathcal{G}, Y, T)$ takes for some particularly simple graphs $Y$.

### 2.2.1 Segment

Suppose $Y$ is a segment:




Let the groups of $\mathcal{G}$ be $G_P, G_Q, G_y$. There is a unique maximal tree $T$, which is the whole $Y$. Then $\pi_1(\mathcal{G}, Y, Y)$ is generated by $G_P$ and $G_Q$, with the relations $a^y = a^{\bar{y}}$ for all $a \in G_y$.

Then the fundamental group is the amalgamated free product $\pi_1(\mathcal{G}, Y, T) = G_P *_{G_y} G_Q$.

*Link to topology:*
Consider spaces $X_P, X_Q, X_y$ with fundamental groups $G_P, G_Q, G_y$ respectively. (Such a space exists for any group $G$, and in fact we can take it to be a CW-complex with all other homotopy groups trivial, in which case it is called an Eilenberg-MacLane space, denoted $K(G, 1)$.) Let the injective homomorphisms in $(\mathcal{G}, Y)$ be $f_P : G_y \hookrightarrow G_P,\ f_Q : G_y \hookrightarrow G_Q$.

Then there are continuous maps $\phi_P : X_y \hookrightarrow X_P,\ \phi_Q : X_y \hookrightarrow X_Q$, inducing the maps $f_P$ and $f_Q$ respectively on the fundamental groups.

We can define a mapping cylinder $M = X_y \times [0, 1]$, and the total space

$$X = (X_P \sqcup M \sqcup X_Q)/\sim \text{ where for } x \in X_y,\ M \ni (x, 0) \sim \phi_P(x) \in X_P,\ M \ni (x, 1) \sim \phi_Q(x) \in X_Q.$$

Then using the Seifert−Van Kampen Theorem,

$$\pi_1(X) = \pi_1(X_P) *_{\pi_1(X_y)} \pi_1(X_Q) = G_P *_{G_y} G_Q = \pi_1(\mathcal{G}, Y, T).$$

This is a simple example of a construction called a graph of spaces, defined by Peter Scott and Terry Wall in [3]. In general, we can define it as follows:

**Definition 9.** For a connected non-empty graph $Y$, a **graph of spaces** $(\mathcal{X}, Y)$ is a collection of spaces $X_P$ for $P \in \mathrm{V}(Y)$, $X_y$ for $y \in \mathrm{E}(Y)$ with $X_y = X_{\bar{y}}$, and continuous maps $\phi_y : X_y \to X_{\mathrm{t}(y)}$.

The corresponding **total space** $X$ is defined as

$$\Big( \bigsqcup_{P \in \mathrm{V}(Y)} X_P \sqcup \bigsqcup_{y \in \mathrm{E}(Y)} (X_y \times [0,1]) \Big) \Big/ \sim$$

where $\sim$ is the minimal equivalence relation where for all $y \in \mathrm{E}(Y)$:

$$\begin{aligned} X_y \times [0,1] \ni (x,t) &\sim (x, 1-t) \in X_{\bar{y}} \times [0,1] && \text{for } x \in X_y,\ t \in [0,1], \\ X_y \times [0,1] \ni (x,1) &\sim \phi_y(x) \in X_{\mathrm{t}(y)} && \text{for } x \in X_y. \end{aligned}$$

From a graph of spaces $(\mathcal{X}, Y)$ we can construct a graph of groups $(\mathcal{G}, Y)$ with the same underlying graph $Y$ by taking fundamental groups of the spaces. (Note though that in general we will not get injective homomorphisms from the edge groups into vertex groups, but we can define the fundamental group of a graph of groups without this condition.) There is a general result that states that if $X$ is the total space of $(\mathcal{X}, Y)$, then $\pi_1(X) \cong \pi_1(\mathcal{G}, Y)$.

From this topological perspective we can see the reason for the relation $ya^y\bar{y} = a^{\bar{y}}$ in $F(\mathcal{G}, Y)$. $a \in G_y$ corresponds to a loop in $X_y$, say based at $P_0$. Then $a^y$ corresponds to this loop in $X_y \times \{1\}$, whereas $a^{\bar{y}}$ corresponds to the loop in $X_y \times \{0\}$. Now $y$ corresponds to paths from $X_y \times \{0\}$ to $X_y \times \{1\}$, say the path $t \mapsto (P_0, t)$. Then $ya^y\bar{y}$ is a loop based at $(P_0, 0)$, and it is homotopic in $X_y$ to the loop $a^{\bar{y}}$. Hence for the isomorphism between $\pi_1(X)$ and $\pi_1(\mathcal{G}, Y)$ to work, we need $ya^y\bar{y} = a^{\bar{y}}$ to hold in $\pi_1(\mathcal{G}, Y)$.

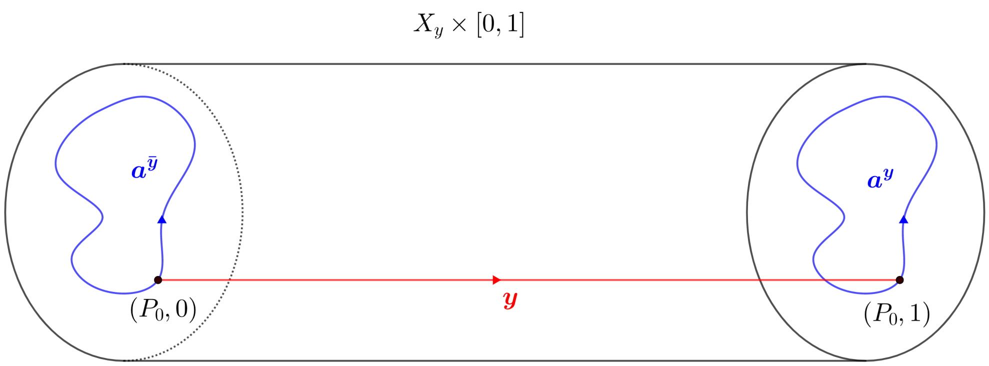



Figure 5: The paths corresponding to $a^{\bar{y}}$ and $ya^y\bar{y}$ are homotopic in the mapping cylinder $X_y \times [0,1]$.

### 2.2.2 Tree

Now we will consider the case where $Y$ is a general tree. There is still a unique maximal tree, which is $Y$.

**Definition 10.** Consider a family of groups indexed by a set $I$, $\{G_i : i \in I\}$, and for all $i, j$ a (possibly empty) set of homomorphisms $F_{ij} \subset \mathrm{Hom}(G_i, G_j)$. Then the **direct limit** of this system is a group $G$ with homomorphisms $\phi_i : G_i \to G$ for all $i \in I$ such that $\forall i, j \in I,\ \forall f_{ij} \in F_{ij},\ \phi_j \circ f_{ij} = \phi_i$, and satisfying the following universal property:

For any group $H$ with homomorphisms $\psi_i : G_i \to H$ such that $\forall i, j \in I,\ \forall f_{ij} \in F_{ij},\ \psi_j \circ f_{ij} = \psi_i$, there exists a unique homomorphism $h : G \to H$ such that $\forall i \in I,\ \psi_i = h \circ \phi_i$.

Thus $G$ is such that the inner triangle commutes in the diagram below, and for any $H$ making the outer triangle commute, there is a unique homomorphism $h : G \to H$ making the entire diagram commute:

$$\begin{tikzcd}
G_i \arrow[rr, "f_{ij}"] \arrow[dr, "\phi_i"] \arrow[ddr, bend right, "\psi_i"'] & & G_j \arrow[dl, "\phi_j"'] \arrow[ddl, bend left, "\psi_j"] \\
& G \arrow[d, dashed, "h"] & \\
& H &
\end{tikzcd}$$

Figure 6: Direct limit

A simple case of a direct limit is that of three groups $A$, $B$, $C$, with injective homomorphisms $C \hookrightarrow A$, $C \hookrightarrow B$. Then the direct limit of this system is the amalgamated free product $A *_C B$.

In general, the direct limit is generated by the $G_i$, with additional relations given by

$$x = f_{ij}(x) \text{ for all } i, j \in I,\ x \in G_i,\ f_{ij} \in F_{ij}.$$

Hence if we take the groups $G_P$ for $P \in \mathrm{V}(Y)$, $G_y$ for $y \in \mathrm{E}(Y)$, and the homomorphisms $G_y \hookrightarrow G_{\mathrm{t}(y)}$ given by $a \mapsto a^y$, then the direct limit of this system is generated by the $G_P$ and $G_y$ subject to

$$a = a^y = a^{\bar{y}} \text{ for all } y \in \mathrm{E}(Y),\ a \in G_y.$$

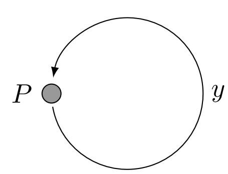
Then it is equal to the group generated by the $G_P$ subject to

$$a^y = a^{\bar{y}} \text{ for all } y \in \mathrm{E}(Y),\ a \in G_y,$$

which is precisely $\pi_1(\mathcal{G}, Y, Y)$.

### 2.2.3 Loop

Now consider the case of $Y$ having a single vertex and one loop:




Here the maximal tree $T$ is the single vertex $P$, and so $\pi_1(\mathcal{G}, Y, T) = F(\mathcal{G}, Y)$ is generated by $G_P$ and $y$, with the relations

$$y a^y y^{-1} = a^{\bar{y}} \text{ for all } a \in G_y.$$

This is known as an HNN extension (which stands for Higman, Neumann, Neumann, who first introduced them in 1949):

**Definition 11.** Consider a group $A$ with two isomorphic subgroups, $C$ and $C'$, and isomorphism $\phi : C \xrightarrow{\sim} C'$. Then the **HNN extension** $A*_C$ is generated by $A$ and an extra element $s$, with the relations

$$sgs^{-1} = \phi(g) \text{ for all } g \in C.$$

Thus $C$ and $C'$ are conjugate in $A*_C$.

There is also an alternate definition: for $n \in \mathbb{Z}$ let $A_n$ be a copy of $A$, $C_n$ a copy of $C$. Let $H$ be the direct limit of the following diagram:

$$\begin{tikzcd}
 & A_{n-1} & & A_n & & A_{n+1} & \\
\cdots & & C_{n-1} \arrow[ul, "\phi"] \arrow[ur, "\mathrm{id}"'] & & C_n \arrow[ul, "\phi"] \arrow[ur, "\mathrm{id}"'] & & \cdots
\end{tikzcd}$$

Then there is a shift map $u : H \to H$ induced by the canonical isomorphisms $\mathrm{id} : A_n \to A_{n+1}$. Then let $G$ be the semi-direct product of $H$ with $\mathbb{Z} = \langle s \rangle$, with $s$ acting by $u$.

Identifying $A$ with $A_0$, for any $g \in C \leqslant A$, $u(g)$ is the image of $g \in C_0$ under $\mathrm{id} : C_0 \to A_1$. But by definition of the direct limit, for $g \in C_0$, $A_0 \ni \phi(g) = \mathrm{id}(g) \in A_1$ in $H$, so $u(g) = \phi(g)$ in $H$.

As $shs^{-1} = u(h)$ for all $h \in H$, in particular for $g \in C$, $sgs^{-1} = u(g) = \phi(g)$.

Since $A_n = s^n A_0 s^{-n}$, we see that $G$ is generated by $A$ and $s$, and the additional relations are $sgs^{-1} = u(g) = \phi(g)$ for $g \in C$. Hence we see that $G = A*_C$ as defined originally.

*Examples:*

(i.) Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, generated by $a, b$. Then $\phi : \langle a \rangle \ \to \ \langle b \rangle$ given by $a \mapsto b$ is an isomorphism of the two subgroups. Then

$$G*_{\langle a \rangle} \ = \ \langle a, b, s \,|\, a^2 = b^2 = abab = sas^{-1}b = 1 \rangle \ = \ \langle a, s \,|\, a^2 = (asas^{-1})^2 = 1 \rangle.$$

Using the alternative definition: let $G_n$ and $A_n$ be copies of $G$ and $\langle a \rangle$ respectively, with $G_n$ generated by $a_n, b_n$, and $A_n$ by $a'_n$.
Then $H$ is generated by $\{a_n, b_n : n \in \mathbb{Z}\}$, and as $G_n \ni \phi(a'_n) = \mathrm{id}(a'_n) \in G_{n+1}$ in $H$, $b_n = a_{n+1}$.

Thus $H = \langle a_n : n \in \mathbb{Z} \,|\, a_n^2 = (a_n a_{n+1})^2 = 1 \rangle$.

The shift map $u$ is $a_n \mapsto a_{n+1}$, so there is a map $\psi : \langle s \rangle \to \mathrm{Aut}(H)$ given by $s \mapsto u$.

Then $G*_{\langle a \rangle} = H \rtimes_\psi \langle s \rangle$ is generated by $H$ and $\langle s \rangle$ with the additional relations $a_{n+1} = s a_n s^{-1}$. Hence, writing $a$ for $a_0$,

$$G*_{\langle a \rangle} \ = \ \langle a, s \,|\, a^2 = (asas^{-1})^2 = 1 \rangle.$$

(ii.) Let $A = \mathbb{Z} = \langle a \rangle$. Then $A$ has infinitely many subgroups isomorphic to $A$. For example, we can take $C$ to be $A$, and $C' = \langle a^{-1} \rangle$, i.e. we consider the automorphism of $A$ given by $a \mapsto a^{-1}$.

Then $A*_C = \langle a, s | sas^{-1} = a^{-1} \rangle$. This is a group known as the Baumslag-Solitar group $\mathrm{BS}(1, -1)$. (In general, $\mathrm{BS}(n, m) = \langle a, b \,|\, ba^n = a^m b \rangle$ and arises as the HNN extension $\mathbb{Z}*_{n\mathbb{Z}}$, with the subgroup isomorphism $\phi : n\mathbb{Z} \xrightarrow{\sim} m\mathbb{Z}$.)

$\mathrm{BS}(1, -1)$ is the fundamental group of the Klein bottle, and we can see that this is an example of the topological correspondence described in 2.2.1.

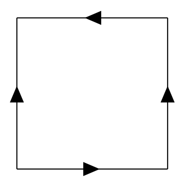
The graph of groups and corresponding graph of spaces are:


Figure 7: Graph of groups and graph of spaces for the HNN extension $\mathrm{BS}(1, -1)$. (We treat $S^1$ as the unit interval with the endpoints identified.)

Then identifying the vertex space $S^1$ with one end of the mapping cylinder $S^1 \times [0, 1]$ corresponding to the loop, the total space is

$$\left(S^1 \times [0, 1]\right) / \sim \ \text{, where } (x, 0) \sim (1 - x, 1) \text{ for all } x \in S^1.$$

But this is precisely the Klein bottle obtained from the standard gluing of the unit square:



## 2.3 Reduced words

### 2.3.1 Definition and results

Consider a word $(c, \mu)$ of type $c$ in $(\mathcal{G}, Y)$, with $c = (y_0, ..., y_n)$, $\mu = (r_0, ..., r_n)$. We would like to know whether or not $|c, \mu| = 1$. We know that if $c$ has a backtracking $y_{i+1} = \overline{y_i}$, and $r_i = a^y$ for some $a \in G_y$, then we can replace $y_i r_i y_{i+1} = y a^y \bar{y}$ by $a^{\bar{y}}$ in $|c, \mu|$, thus simplifying it. We will say that a word is reduced if it can not be simplified in this way.

**Definition 12.** A word $(c, \mu)$ of length $n \geqslant 1$ is **reduced** if whenever $y_{i+1} = \overline{y_i}$, $r_i \notin G^{y_i}_{y_i}$.
(Where $G^y_y$ is the image of $G_y$ in $G_{\mathrm{t}(y)}$ under $a \mapsto a^y$.)

A word $(c, \mu)$ of length $n = 0$ is reduced if $r_0 \neq 1$.

In fact it turns out that a word being reduced is enough for its associated element of $F(\mathcal{G}, Y)$ to be non-trivial. This is a tool that will be useful to us in the next section. The proof is the most complicated of any in this essay, and it relies on a technique called *dévissage*, which reduces the graph $Y$ by contracting subgraphs until we obtain either a segment or a loop.

But first, let us state the theorem and a few corollaries:

**Theorem 2.** If $(c, \mu)$ is a reduced word, then $|c, \mu| \neq 1$.

**Corollary 1.** The homomorphisms $G_P \to F(\mathcal{G}, Y)$ are injective for $P \in \mathrm{V}(Y)$.

*Proof.* This is the case where $c$ has length 0, as we get that for $1 \neq r_0 \in G_P$, $r_0 \neq 1$ in $F(\mathcal{G}, Y)$. □

**Corollary 2.** If $(c,\mu)$ is reduced, and if $c$ has length $n \geqslant 1$, then for $P_0 = \mathrm{o}(c)$, $|c,\mu| \notin G_{P_0}$.

*Proof.* Suppose that $|c,\mu| = x \in G_{P_0}$. Then if $\mu' = (x^{-1}r_0, r_1, ..., r_n)$, then $(c,\mu')$ is reduced, but $|c,\mu'| = 1$. $\square$

**Corollary 3.** Let $c$ be a cycle in $Y$, $(c,\mu)$ a word of type $c$. Then for any maximal tree $T$ of $Y$, the image of $|c,\mu|$ under the projection into $\pi_1(\mathcal{G},Y,T)$ is $\neq 1$.

*Proof.* If $P_0 = \mathrm{o}(c) = \mathrm{t}(c)$, then $|c,\mu| \in \pi_1(\mathcal{G},Y,P_0)$. But we know from Prop. 4 that the projection $F(\mathcal{G},Y) \to \pi_1(\mathcal{G},Y,T)$ restricts to an isomorphism on $\pi_1(\mathcal{G},Y,P_0)$. Hence as $|c,\mu| \neq 1$, its image is also non-trivial. $\square$

### 2.3.2 The dévissage argument

Let $Y'$ be a non-empty connected subgraph of $Y$. Then $(\mathcal{G},Y)$ restricts to a graph of groups $(\mathcal{G}|_{Y'},Y')$.

Suppose that Theorem 2 holds for $(\mathcal{G}|_{Y'},Y')$. Our aim is to show that this implies that the theorem also holds for $(\mathcal{G},Y)$.

We can define the contraction $W = Y/Y'$, with:

$$\begin{aligned}
&\mathrm{V}(W) = \mathrm{V}(Y) \setminus \mathrm{V}(Y') \ \cup\ \{(Y')\},\\
&\mathrm{E}(W) = \mathrm{E}(Y) \setminus \mathrm{E}(Y'),\\
&{}^{-} : \mathrm{E}(W) \to \mathrm{E}(W) \ \text{ the restriction of } \ {}^{-} : \mathrm{E}(Y) \to \mathrm{E}(Y),\\
&\mathrm{o},\mathrm{t} : \mathrm{E}(W) \to \mathrm{V}(W) \ \text{ induced from } Y \text{ by the projection } \ \mathrm{V}(Y) \to \mathrm{V}(W).
\end{aligned}$$

Then define the graph of groups $(\mathcal{H},W)$ with:

$$\begin{aligned}
&H_P = G_P \quad \text{for } P \in \mathrm{V}(Y) \setminus \mathrm{V}(Y'),\\
&H_{(Y')} = F(\mathcal{G}|_{Y'},Y'),\\
&H_y = G_y \quad \text{for } y \in \mathrm{E}(W),\\
&H_y \to H_{\mathrm{t}(y)} \text{ induced from } Y.
\end{aligned}$$

Note that since Theorem 2 holds for $(\mathcal{G}|_{Y'},Y')$, by Corollary 1, the maps $G_P \to H_{(Y')} = F(\mathcal{G}|_{Y'},Y')$ are injective for $P \in \mathrm{V}(Y')$.
Hence the homomorphisms $H_y \to H_{\mathrm{t}(y)}$ are indeed injective for all edges $y$.

There is a natural homomorphism $F(\mathcal{G},Y) \to F(\mathcal{H},W)$ induced by the projection of the graphs of groups.

**Lemma 1.** The homomorphism $F(\mathcal{G},Y) \to F(\mathcal{H},W)$ is an isomorphism.

*Proof.* It is clear that the generators of the two groups are the same, with some of the generators of $F(\mathcal{G},Y)$ simply grouped under the name $H_{(Y')}$ in $F(\mathcal{H},W)$. The relations are also the same, the only case where this does not follow directly from the definition is for edges $y$ of $W$ which have $(Y')$ as a vertex.

Suppose $y \in \mathrm{E}(W)$, and in $W$ $\mathrm{t}(y) = (Y')$. Then in $Y$ $\mathrm{t}(y) = P$ for some $P \in \mathrm{V}(Y')$. Then the image of $H_y$ under $a \to a^y$ can be identified with the image of $G_y$ in $G_P$. (Again we use Corollary 1 for $Y'$, allowing us to consider $G_P$ as a subgroup of $H_{(Y')}$.) Hence we have canonical isomorphisms $H_y^y \to G_y^y$ for $y \in \mathrm{E}(W)$, and so the relations

$$y a^y \bar{y} = a^{\bar{y}} \text{ for } a \in G_y = H_y$$

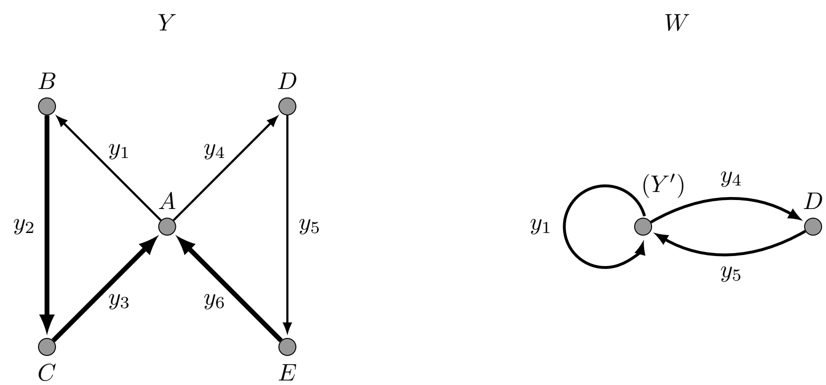
are exactly the same in $F(\mathcal{G},Y)$ and $F(\mathcal{H},W)$. $\square$

For a word $(c, \mu)$ in $(\mathcal{G}, Y)$ we can define the corresponding word $(c', \mu')$ in $(\mathcal{H}, W)$:

- If $c$ lies entirely in $Y'$, then $c'$ is the empty path and $\mu' = (|c, \mu|)$.
- If $c = (y_1, ..., y_n)$ contains a subpath $\hat{c} = (y_k, ..., y_m)$ entirely in $Y'$, then replace $c$ by $(y_1, ..., y_{k-1}, y_{m+1}, ..., y_n)$ and $\mu$ by $(r_0, ..., r_{k-2}, |\hat{c}, \hat{\mu}|, r_{m+1}, ..., r_n)$, where $\hat{\mu} = (r_k, ..., r_{m+1})$.
  Repeat this until $c$ lies entirely in $\mathrm{E}(Y) \setminus \mathrm{E}(Y')$.

*Example:*

In the graph below, let $Y'$ be the subgraph with edges $y_2, y_3, y_6$ (and their inverses).
Let $c$ be the figure of eight loop $(y_1, y_2, y_3, y_4, y_5, y_6)$. Then $c' = (y_1, y_4, y_5)$ and $\mu' = (r_0, r_1 y_2 r_2 y_3 r_3, r_4, r_5 y_6 r_6)$.




**Lemma 2.** If $(c, \mu)$ is reduced, so is $(c', \mu')$.

*Proof.* If $c'$ has length 0, then $c$ lies inside $Y'$. As we assumed that Theorem 2 holds for $(\mathcal{G}|_{Y'}, Y')$, $|c, \mu| \neq 1$ in $F(\mathcal{G}|_{Y'}, Y')$. Then $\mu' = (q_0)$, where $q_0 = |c, \mu| \neq 1$, so $(c', \mu')$ is reduced.

Now suppose $c'$ has length $m \geqslant 1$. Let $c' = (w_1, ..., w_m), \mu' = (q_0, ..., q_m)$. Suppose $c'$ has a backtracking $w_{i+1} = \overline{w_i}$. If $\mathrm{t}(w_i) \neq (Y')$, this backtracking is not an issue as $c$ is reduced.

If $\mathrm{t}(w_i) = (Y')$, there are two cases:

(i.) If this bactracking exists in $c$, i.e. for some $j$, $w_i = y_j,\ w_{i+1} = y_{j+1},\ q_{i+1} = r_{j+1}$, then since we can identify $H_{w_i}^{w_i}$ with $G_{y_j}^{y_j}$ (cf. proof of Lemma 1), $r_{j+1} \notin G_{y_j}^{y_j}$ implies $q_{i+1} \notin H_{w_i}^{w_i}$.

(ii.) If this backtracking comes from a cycle of $c$ in $Y'$, i.e. for some $j < k$,
$w_i = y_j,\ w_{i+1} = y_{k+1},\ q_{i+1} = r_{j+1} y_{j+1} ... y_k r_{k+1}$, then we use Corollary 2.

Let $\hat{c} = (y_{j+1}, ..., y_k)$, $\hat{\mu} = (r_{j+1}, ..., r_{k+1})$. Then $\hat{c}$ is a cycle based at $P = \mathrm{t}(y_j)$. So by Corollary 2, $|\hat{c}, \hat{\mu}| \notin G_P$, as $(\hat{c}, \hat{\mu})$ is reduced, being a subword of $(c, \mu)$.

But $q_{i+1} = |\hat{c}, \hat{\mu}|$, and identifying $H_{w_i}^{w_i}$ with $G_{y_j}^{y_j} \leqslant G_P$, we see that $q_{i+1} \notin H_{w_i}^{w_i}$. □

Combining Lemmas 1 and 2 we obtain the following result:

**Lemma 3.** Suppose $Y'$ is a connected non-empty subgraph of $Y$, $W = Y/Y'$, $(\mathcal{H}, W)$ the graph of groups defined as above. Then if Theorem 2 holds for both $(\mathcal{G}|_{Y'}, Y')$ and $(\mathcal{H}, W)$, then it also holds for $(\mathcal{G}, Y)$.

*Proof.* Take a reduced word $(c, \mu)$ in $(\mathcal{G}, Y)$. By Lemma 2, the corresponding word $(c', \mu')$ of $(\mathcal{H}, W)$ is reduced, so $|c', \mu'| \neq 1$ (by Theorem 2 applied to $(\mathcal{H}, W)$).
By Lemma 1, we have the isomorphism $F(\mathcal{H}, W) \to F(\mathcal{G}, Y)$, which maps $|c', \mu'|$ to $|c, \mu|$. Hence $|c, \mu| \neq 1$. □

### 2.3.3 Normal form for free products with amalgamation

We need one more ingredient for the proof of Theorem 2, which is a normal form for free products with amalgamation.

Let $G = A *_C B$, and let $S, T$ be right transversals of $C$ in $A, B$ respectively, with the representatives of the coset $C$ removed.

Let $W$ be the set of finite alternating words in the letters $a, b$ (such as $a$, $baba$ or the empty word 0).
For $w \in W$ define $\Pi(w)$ as the corresponding Cartesian product of copies of $S$ and $T$, with
$\Pi(0) = \emptyset,\ \Pi(a) = S,\ \Pi(b) = T$, and $\Pi(w_1 w_2) = \Pi(w_1) \times \Pi(w_2)$ for any words $w_1, w_2 \in W$.

Let $G' = \bigsqcup_{w \in W} C \times \Pi(w)$. (So elements of $G'$ look like one of $c$, $cs_1t_1...s_nt_n$, $cs_1t_1...s_n$, $ct_1s_1...t_ns_n$, $ct_1s_1...t_n$, with $c \in C,\ s_i \in S,\ t_i \in T$.)
Then $G'$ is a group under the operation of concatenation, with relations from $A$ and $B$, as we can 'slide' the elements of $C$ through to the left – e.g. if $s \in S, c \in C$, then $sc \in A$, so in $A$, $sc = c's'$ for some $c' \in C, s' \in S$.

We will show that in fact $G'$ is $G$, by showing that it satisfies the universal property for $G$. Let $f_A : A \to G'$ be the homomorphism given by the unique representation of $a \in A$ as $a = cs$ with $c \in C,\ s \in S$. Similarly define $f_B : B \to G'$.

Suppose $H$ is a group with homomorphisms $\phi_A, \phi_B : A, B \to H$ such that $\phi_A(c) = \phi_B(c)$ for $c \in C$. Then if a homomorphism $h : G' \to H$ exists with $h \circ f_A = \phi_A,\ h \circ f_B = \phi_B$, we must have:

$$\begin{aligned} h(c) &= \phi_A(c) = \phi_B(c) && \text{for } c \in C \\ h(s) &= \phi_A(s) && \text{for } s \in S \\ h(t) &= \phi_B(t) && \text{for } t \in T. \end{aligned}$$

Hence such an $h$ is unique. and we can check that $h$ is in fact a homomorphism. From the definition of $h$ it is clear that for $c \in C, s \in S,\ t \in T$, $h(cs) = h(c)h(s), h(ct) = h(c)h(t), h(st) = h(s)h(t), h(ts) = h(t)h(s)$.

So the only thing we need to check is that $h(sc) = h(s)h(c)$ and $h(tc) = h(t)h(c)$. But if $sc = c's'$ in $A$, then

$$h(sc) = h(c')h(s') = \psi_A(c')\psi_A(s') = \psi_A(c's') = \psi_A(sc) = \psi_A(s)\psi_A(c) = h(s)h(c).$$

Similarly we show that $h(tc) = h(t)h(c)$, and so we are done.

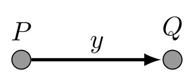
Thus we have proved:

**Proposition 5.** Every element of $A *_C B$ can be uniquely written in one of the following forms:
$c$, $cs_1t_1...s_nt_n$, $cs_1t_1...s_n$, $ct_1s_1...t_ns_n$ or $ct_1s_1...t_n$, with $c \in C,\ s_i \in S,\ t_i \in T$.

Note that this implies that the homomorphisms $f_A, f_B : A, B \to A *_C B$ are injective.

### 2.3.4 Proof of Theorem 2

Case (i): $Y$ is a segment.




Let $A = G_P, B = G_Q, C = G_y$.
First consider the case $n = 0$. we know from Prop. 5 that $A, B$ inject into $A *_C B = \pi_1(\mathcal{G}, Y, Y)$. Then since $\pi_1(\mathcal{G}, Y, Y) \cong \pi_1(\mathcal{G}, Y, P)$ and $\pi_1(\mathcal{G}, Y, P) \leqslant F(\mathcal{G}, Y)$, we see that $G_P$ and $G_Q$ inject into $F(\mathcal{G}, Y)$, so in particular the case of $n = 0$ holds.

Now suppose $n \geqslant 1$. By projecting $F(\mathcal{G}, Y)$ onto $\langle y \rangle$, we see that $|c, \mu|$ can only be 1 if $n$ is even, so assume that $n \geqslant 2$.

Now project $|c, \mu|$ onto $\pi_1(\mathcal{G}, Y, Y) \cong A *_C B$ to get $r_0r_1...r_n$. The $r_i$ are alternately from $A$ and $B$, and since $(c, \mu)$ is reduced, $r_i \notin C$ for $i = 1, ..., n-1$.

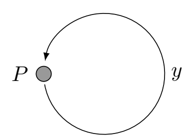
Now replacing $r_i$ by $c_i s_i$ or $c_i t_i$ with $c_i \in C, s_i \in S, t_i \in T$, and sliding the $c_i$ through to the left, we obtain the normal form for $r_0 r_1 ... r_n$, which contains at least $n+1-2 \geqslant 1$ elements of $S$ or $T$. (The -2 comes from the possibility that $r_0, r_n \in C$.)

Hence by Prop. 5, $r_0 r_1 ... r_n \neq 1$, and hence $|c, \mu| \neq 1$, and so we are done.

Case (ii): $Y$ is a tree.
It is enough to prove the theorem for all finite trees, since every path in $Y$ is contained in a finite subtree.

Hence we can induct on the number of edges, $e = \frac{1}{2}|\mathrm{E}(Y)|$. We proved the base case $e = 1$ above. If $e > 1$, we can contract a subgraph $Y'$ consisting of a single edge of $Y$ to obtain the graph $W$. By Prop. 3, $W$ is again a tree, and it has less edges, so the theorem holds for it by induction. Since the theorem also holds for the segment $Y'$, it follows from Lemma 3 that the theorem holds for $Y$.

Case (iii): $Y$ is a loop.




Let $A = G_P, \ C = G_y$.
Then $F(\mathcal{G}, Y) = A*_C$, with the injective homomorphisms $C \to A$ being $g \to g^y$ and $g \to g^{\bar{y}}$. Let us identify $C$ with its image under $g \to g^y$, and let $g \to g^{\bar{y}}$ be $\phi$. Then, as in 2.2.3, define $H$ as the direct limit of the following diagram, where $C_i$ and $A_i$ are copies of $C$ and $A$ for $i \in \mathbb{Z}$:

$$\begin{tikzcd}
\cdots & A_{n-1} & & A_n & & A_{n+1} & \cdots \\
& & C_{n-1} \arrow[ul, "\phi"] \arrow[ur, "\mathrm{id}"'] & & C_n \arrow[ul, "\phi"] \arrow[ur, "\mathrm{id}"'] & &
\end{tikzcd}$$

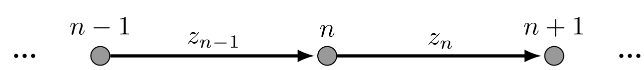
Then $A*_C = H \rtimes_\psi \langle y \rangle$ where $\psi : y \mapsto u$ for the shift map $u$. Note that $A_n = y^n A_0 y^{-n}$.

For a reduced word $(c, \mu)$, $|c, \mu| = r_0 y^{e_1} r_1 y^{e_2} r_2 ... y^{e_n} r_n$, where $r_i \in A$, $e_i = \pm 1$, and if $e_{i+1} = -e_i$, then $r_i \notin C^{y^{e_i}}$. (Note $C^y = C, \ C^{y^{-1}} = \phi(C)$.)

Now, $H$ is the normal subgroup of $A*_C$ which is the kernel of the projection onto $\langle y \rangle$, so if $\sum e_i \neq 0$, then $|c, \mu| \notin H$, hence $|c, \mu| \neq 1$.

Suppose then that $\sum e_i = 0$. Let $d_i = e_1 + ... + e_i$, and let $s_i = y^{d_i} r_i y^{-d_i}$.
Then $|c, \mu| = s_0 s_1 ... s_n$, with $s_i \in A_{d_i}$, $d_0 = d_n = 0$, $d_{i+1} - d_i = e_{i+1} = \pm 1$, and if $d_{i+1} = d_{i-1}$, then $s_i \notin y^{d_i} C^{y^{e_i}} y^{-d_i}$.

Now let $T$ be the tree with vertices labelled by $\mathbb{Z}$, and edges $z_n$ from $n$ to $n+1$ for all $n \in \mathbb{Z}$. Let $(\mathcal{K}, T)$ be the graph of groups with vertex groups $K_n = A_n$, edge groups $K_{z_n} = C_n$, and homomorphisms id and $\phi$:




$$A_{n-1} \xleftarrow{\phi} C_{n-1} \xrightarrow{\mathrm{id}} A_n \xleftarrow{\phi} C_n \xrightarrow{\mathrm{id}} A_{n+1}$$

Figure 8: The graph of groups $(\mathcal{K}, T)$.

Then $H = \pi_1(\mathcal{K}, T, T)$ (see 2.2.2).
Let $c'$ be the path $(w_1, .., w_n)$, where $w_i$ is the edge from $d_{i-1}$ to $d_i$. (Thus if $e_i = 1$, then $w_i = z_{d_{i-1}}$, and if $e_i = -1$, then $w_i = \overline{z_{d_i}}$.)

Then if $\mu' = (s_0, s_1, ..., s_n)$, since $s_i \in A_{d_i} = K_{d_i}$, we have that $(c', \mu')$ is a word of type $c'$ in $F(\mathcal{K}, T)$. $c'$ is a cycle, as $d_n = d_0 = 0$. Now, $T$ is a tree, so we can apply Theorem 2 to it, and by Corollary 3, the projection of $|c', \mu'|$ into $\pi_1(\mathcal{K}, T, T)$ is non-trivial. But this projection is precisely $s_0 s_1 ... s_n = |c, \mu|$. Thus we have that $|c, \mu|$ is non-trivial in $\pi_1(\mathcal{K}, T, T) = H$, hence it is non-trivial in $F(\mathcal{G}, Y)$.

Case (iv): $Y$ is any graph.
As in case (ii), we only need to prove this for finite $Y$. Again we use induction on $e = \frac{1}{2}|\mathrm{E}(Y)$. The base case $e = 1$ is either a segment or a loop, and we have proved both these cases already.

For $e > 1$, we can contract a segment or loop $Y'$ to get a graph $W$ with less edges. The theorem holds for $Y'$, and for $W$ by induction, and thus by Lemma 3 it holds for $Y$ so we are done. □

# 3 The main theorem

We have the following result from topology:

**Proposition 6.** Let $\tilde{X}$ be a simply-connected Hausdorff space, and let $G$ be a group acting on $\tilde{X}$ freely and properly discontinuously by homeomorphisms. Then if $X = G \backslash \tilde{X}$, the fundamental group of $X$ is $G$, and $\tilde{X}$ is the universal cover of $X$.

The main theorem of Bass-Serre Theory is a stronger analogue of this statement for graphs. Due to their discrete nature, any action on graphs is automatically properly discontinuous, so we don't need to worry about that condition. But by working with graphs of groups rather than simply graphs, we can keep track of stabilisers in our action, and hence we can consider actions which are not free.

## 3.1 The Bass-Serre tree: from graph of groups to action on graph

Here we will define the Bass-Serre tree, which is essentially a universal cover of a graph of groups.

We take a graph of groups $(\mathcal{G}, Y)$, with $Y$ connected and non-empty. Pick an orientation $E_+$ of $Y$, and define a sign function $\sigma$ on $\mathrm{E}(Y)$, with:

$$\sigma(y) = \begin{cases} 0 & \text{if } y \in E_+ \\ 1 & \text{if } y \notin E_+ \end{cases}$$

Let $|y|$ be the edge out of $\{y, \bar{y}\}$ which belongs to $E_+$.

We will construct the following objects:

- a graph $\tilde{X} = \tilde{X}(\mathcal{G}, Y, T)$ [the universal cover]
- an action of $\pi = \pi_1(\mathcal{G}, Y, T)$ on $\tilde{X}$ [the deck transformations]
- a map $p : \tilde{X} \to Y$ inducing an isomorphism of graphs $\pi \backslash \tilde{X} \xrightarrow{\sim} Y$ [the covering map]
- sections $\mathrm{V}(Y) \to \mathrm{V}(\tilde{X})$ and $\mathrm{E}(Y) \to \mathrm{E}(\tilde{X})$, denoted by $P \mapsto \tilde{P}$ and $y \mapsto \tilde{y}$ (i.e. maps such that $p(\tilde{P}) = P$, $p(\tilde{y}) = y$) such that:
  - for $P \in \mathrm{V}(Y)$, the stabiliser $\pi_{\tilde{P}}$ of $\tilde{P}$ is $G_P$
  - for $y \in \mathrm{E}(Y)$, the stabiliser $\pi_{\tilde{y}}$ of $\tilde{y}$ is $G_w^w$ where $w = \overline{|y|}$

(Where does the condition $\pi_{\tilde{y}} = G_w^w$ come from? It is natural to ask for $\pi_{\tilde{y}}$ to be an image of $G_y$, so it should be either $G_y^y$ or $G_{\bar{y}}^{\bar{y}}$. We then link this choice of two options to the orientation $E_+$ to allow us to keep track of it using $\sigma$, and arbitrarily choose the stabiliser to be $G_w^w$ rather than $G_{\bar{w}}^{\bar{w}}$.)

Then the vertices of $\tilde{X}$ must be

$$\mathrm{V}(\tilde{X}) = \bigsqcup_{P \in \mathrm{V}(Y)} \pi \cdot \tilde{P} = \bigsqcup_{P \in \mathrm{V}(Y)} \pi / \pi_{\tilde{P}},$$

and the edges must be

$$\mathrm{E}(\tilde{X}) = \bigsqcup_{y \in \mathrm{E}(Y)} \pi \cdot \tilde{y} = \bigsqcup_{y \in \mathrm{E}(Y)} \pi / \pi_{\tilde{y}}.$$

We can then let $\tilde{P}$ be $1 \cdot \pi_{\tilde{P}} \in \pi/\pi_{\tilde{P}}$, and similarly for $\tilde{y}$.
With these definitions $\pi$ acts naturally on $\tilde{X}$, with $g \cdot h\pi_{\tilde{P}} = gh\pi_{\tilde{P}}$, and similarly for edges.

To complete the definition of $\tilde{X}$ we need to define the reverse map $^-$ and the endpoint maps $\mathrm{o},\ \mathrm{t}$.
It is natural to let $\overline{\tilde{y}} = \tilde{\bar{y}}$.

Now consider $y \in E_+$, with $\mathrm{o}(y) = P,\ \mathrm{t}(y) = Q$. We need to have $\mathrm{o}(\tilde{y}) = a\tilde{P},\ \mathrm{t}(\tilde{y}) = b\tilde{Q}$ for some $a, b \in \pi$. We also need $\pi_{\tilde{y}} = \pi_{\mathrm{o}(\tilde{y})} \cap \pi_{\mathrm{t}(\tilde{y})}$, so

$$G_{\bar{y}}^{\bar{y}} = a\, G_P\, a^{-1} \cap\ b\, G_Q\, b^{-1}. \qquad (*)$$

Since $G_{\bar{y}}^{\bar{y}} \subset G_P$, and $g_y^{-1} G_{\bar{y}}^{\bar{y}} g_y \subset G_Q$, letting $a = 1,\ b = g_y$ makes $(*)$ hold.

Then if instead $y \notin E_+$, with $\mathrm{o}(y) = P,\ \mathrm{t}(y) = Q$ again, we use that $\bar{y} \in E_+$ to see that

$$\begin{aligned} \mathrm{o}(y) &= \mathrm{t}(\bar{y}) = g_{\bar{y}}\tilde{Q} = g_y^{-1}\tilde{Q}, \\ \mathrm{t}(y) &= \mathrm{o}(\bar{y}) = \tilde{P}. \end{aligned}$$

Hence we define for all $y \in \mathrm{E}(Y),\ g \in \pi$:

- $\overline{g\tilde{y}} = g\tilde{\bar{y}}$
- $\mathrm{o}(g\tilde{y}) = g g_y^{-\sigma(y)} \widetilde{\mathrm{o}(y)}$
- $\mathrm{t}(g\tilde{y}) = g g_y^{1-\sigma(y)} \widetilde{\mathrm{t}(y)}$

This is well-defined: the reverse map is an involution and $\mathrm{o}(z) = \mathrm{t}(\bar{z})$ for $z \in \mathrm{E}(\tilde{X})$.

Let us also check that the edge stabilisers are as required: We need to show that for any $h \in \pi_{\tilde{y}}$, $h\mathrm{o}(\tilde{y}) = \mathrm{o}(\tilde{y})$ and $h\mathrm{t}(\tilde{y}) = \mathrm{t}(\tilde{y})$.

Note for $y \in \mathrm{E}(Y)$,

$$\pi_{\tilde{y}} = g_y^{1-\sigma(y)}\, G_y^y\, g_y^{\sigma(y)-1} = g_y^{-\sigma(y)}\, G_{\bar{y}}^{\bar{y}}\, g_y^{\sigma(y)}.$$

(This is easy to check considering the two cases $\sigma(y) = 0, 1$.)

Hence

$$g_y^{\sigma(y)}\, \pi_{\tilde{y}}\, g_y^{-\sigma(y)} = G_{\bar{y}}^{\bar{y}} \subset G_{\mathrm{o}(y)} = \pi_{\widetilde{\mathrm{o}(y)}},$$

so

$$\forall h \in \pi_{\tilde{y}},\ h g_y^{-\sigma(y)} \widetilde{\mathrm{o}(y)} = g_y^{-\sigma(y)} \widetilde{\mathrm{o}(y)}, \quad \text{hence } h\mathrm{o}(\tilde{y}) = \mathrm{o}(\tilde{y}).$$

Similarly $\forall h \in \pi_{\tilde{y}},\ h\mathrm{t}(\tilde{y}) = \mathrm{t}(\tilde{y})$, and hence the graph $\tilde{X}$ is well-defined.

Note that if $g_y = 1$, then $\tilde{y}$ is a lift of $y$, and in particular $T$ lifts to a tree $\tilde{T}$ in $\tilde{X}$.

**Theorem 3.** Let $(\mathcal{G}, Y)$ be a graph of groups with $Y$ connected and non-empty, let $T$ be a maximal tree of $Y$, $E_+$ an orientation of $Y$. Then the graph $\tilde{X} = \tilde{X}(\mathcal{G}, Y, T)$ is a tree.

*Proof.*

1.) $\tilde{X}$ is connected:
If $y \in \mathrm{E}(Y)$, then either $\mathrm{o}(\tilde{y}) = \widetilde{\mathrm{o}(y)}$ or $\mathrm{t}(\tilde{y}) = \widetilde{\mathrm{t}(y)}$, so one end of $\tilde{y}$ lies in $\tilde{T}$. Hence if $W$ is the smallest subgraph of $\tilde{X}$ which contains $\tilde{y}$ for all $y \in \mathrm{E}(Y)$, then $W$ is connected.
Since $\mathrm{E}(\tilde{X}) = \pi \cdot \mathrm{E}(Y)$, we have $\tilde{X} = \pi \cdot W$.

Thus it suffices to find a generating set $S$ of $\pi$ such that for any $s \in S \cup S^{-1}$, $W \cup sW$ is connected. Indeed, then for any $g \in \pi$, we can write $g = s_1 s_2 ... s_n$ for $s_i \in S \cup S^{-1}$. Then for any $i$, $W \cup s_i W$ is connected, so $s_1 ... s_{i-1} W \cup s_1 ... s_{i-1} s_i W$ is connected. Hence $W \cup s_1 W \cup s_1 s_2 W \cup ... \cup gW$ is connected, hence $W$ and $gW$ lie in the same connected component of $\tilde{X}$. As this holds for any $g \in \pi$, this implies that $\tilde{X}$ is connected.

Take

$$S = \bigcup_{P \in \mathrm{V}(Y)} G_P \ \cup \{g_y : y \in \mathrm{E}(Y)\}.$$

Then $S$ generates the image of $F(\mathcal{G}, Y)$ under the projection onto $\pi_1(\mathcal{G}, Y, T) = \pi$, which is all of $\pi$.

If $s \in G_P$, then as $G_P = \pi_{\tilde{P}}$, $s\tilde{P} = \tilde{P}$, hence $\tilde{P}$ is a common vertex of $W$ and $sW$, so $W \cup sW$ is connected.

If $s = g_y$, consider two cases:

(i) If $\sigma(y) = 0$, then $W \ni \mathrm{t}(\tilde{y}) = g_y \widetilde{\mathrm{t}(y)} \in g_y W$.

(ii) If $\sigma(y) = 1$, then $W \ni \widetilde{\mathrm{o}(y)} = g_y g_y^{-1} \widetilde{\mathrm{o}(y)} = \mathrm{o}(g_y \tilde{y}) \in g_y W$.

In either case $W$ and $g_y W$ share a vertex, so $W \cup g_y W$ is connected.

Hence we have shown that $\tilde{X}$ is connected.

2.) $\tilde{X}$ is a tree:
We need to show that $\tilde{X}$ does not contain closed paths of positive length without backtracking.

Suppose that such a path $\tilde{c}$ does exist, and $\tilde{c} = (h_1 \tilde{y}_1, ..., h_n \tilde{y}_n)$ for some $y_i \in \mathrm{E}(Y)$, $h_i \in \pi$. Let $c$ be the projection $p(\tilde{c})$ in $Y$, and let its vertices be $P_0, ..., P_n$. Since $\tilde{c}$ is a cycle, so is $c$, so $P_0 = P_n$.
Then if $\sigma_i = \sigma(y_i)$ and $g_i = g_{y_i}$, for $i = 1, ..., n$ we have (treating the $i$ modulo $n$):

$$\mathrm{t}(h_i \tilde{y}_i) = \mathrm{o}(h_{i+1}\, \tilde{y}_{i+1}), \text{ hence } \ h_i g_i^{1-\sigma_i} \tilde{P}_i = h_{i+1} g_{i+1}^{-\sigma_{i+1}} \tilde{P}_i.$$

Let $q_i = h_i g_i^{-\sigma_i}$, then

$$q_i g_i \tilde{P}_i = q_{i+1} \tilde{P}_i.$$

Hence if $r_i = (q_i g_i)^{-1} q_{i+1}$, then $r_i \in \pi_{\tilde{P}_i} = G_{P_i}$, and

$$g_i r_i = q_i^{-1} q_{i+1}.$$

Taking the product over all $i$, we see that

$$g_1 r_1 g_2 r_2 ... g_n r_n = 1.$$

So if $\mu = (1, r_1, r_2, ..., r_n)$, then $(c, \mu)$ is a word of type $c$ in $F(\mathcal{G}, Y)$, and the image of $|c, \mu|$ in $\pi_1(\mathcal{G}, Y, T)$ is 1. However we will now show that $(c, \mu)$ is reduced, which is a contradiction due to Corollary 3.

Indeed, suppose $y_{i+1} = \bar{y}_i$, then $g_{i+1} = g_i^{-1}$, and $\sigma_{i+1} = 1 - \sigma_i$. Hence

$$r_i = \left((h_i g_i^{-\sigma_i})g_i\right)^{-1}\left(h_{i+1}(g_i^{-1})^{-(1-\sigma_i)}\right) = g_i^{\sigma_i - 1} h_i^{-1} h_{i+1} g_i^{1-\sigma_i}.$$

We wish to show that $r_i \notin G_{y_i}^{y_i}$, i.e. that

$$h_i^{-1} h_{i+1} \notin g_i^{1-\sigma_i} G_{y_i}^{y_i} g_i^{\sigma_i - 1} = \pi_{\tilde{y}_i}.$$

But since $\tilde{c}$ does not have any backtrackings,

$$h_i \tilde{y}_i \neq \overline{h_{i+1}\tilde{y}_{i+1}} = h_{i+1}\tilde{y}_i.$$

This implies that $h_i^{-1}h_{i+1} \notin \pi_{\tilde{y}_i}$, so $r_i \notin G_{y_i}^{y_i}$.

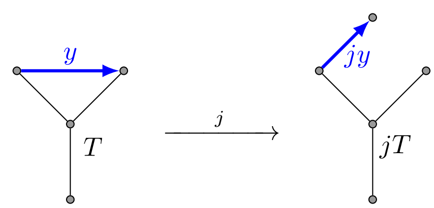
Thus $(c, \mu)$ is reduced, so we got a contradiction, and hence such a path $\tilde{c}$ does not exist. Thus $\tilde{X}$ is a tree. □

## 3.2 From action on graph to graph of groups

Suppose $G$ acts on a connected non-empty graph $X$. Let $Y = G\backslash X$, $T$ a maximal tree of $Y$. We will define a graph of groups $(\mathcal{G}, Y)$ corresponding to this action and we will see that if $X$ is a tree, then $\pi_1(\mathcal{G}, Y, T) \cong G$.

Let us construct $(\mathcal{G}, Y)$:
Take $j : T \to X$ a lift of $T$. Choose an orientation $E_+$ on $Y$ and define $\sigma : \mathrm{E}(Y) \to \{0, 1\}$ as before.

We will extend $j$ to all of $\mathrm{E}(Y)$, such that $\overline{jy} = j\bar{y}$. For this it is enough to define $j$ on $E_+ \setminus \mathrm{E}(T)$. We do this by requiring that for all $y \in E_+$, $\mathrm{o}(jy) \in jT$, hence $\mathrm{o}(jy) = j\mathrm{o}(y)$.




Since both $j\mathrm{t}(y)$ and $\mathrm{t}(jy)$ project to $\mathrm{t}(y)$ in $Y$, there is a $\gamma_y \in G$ such that $\mathrm{t}(jy) = \gamma_y j\mathrm{t}(y)$.
Extend the map $y \mapsto \gamma_y$ to all of $\mathrm{E}(Y)$ by setting $\gamma_{\bar{y}} = \gamma_y^{-1}$ for all $y$ and $\gamma_y = 1$ for $y \in T$.

Now if $\sigma(y) = 1$, then

$$\begin{aligned} \mathrm{o}(jy) &= \mathrm{t}(j\bar{y}) = \gamma_{\bar{y}} j\mathrm{t}(\bar{y}) = \gamma_y^{-1} j\mathrm{o}(y), \\ \mathrm{t}(jy) &= \mathrm{o}(j\bar{y}) = j\mathrm{o}(\bar{y}) = j\mathrm{t}(y). \end{aligned}$$

Hence we see that for all $y$,

$$\begin{aligned} \mathrm{o}(jy) &= \gamma_y^{-\sigma(y)} j\mathrm{o}(y), \\ \mathrm{t}(jy) &= \gamma_y^{1-\sigma(y)} j\mathrm{t}(y). \end{aligned}$$

*Remark.* This is starting to look a lot like the definition of $\tilde{X}$ in the previous section, with $j$ taking the role of the sections $P \mapsto \tilde{P}$, $y \mapsto \tilde{y}$. This is not a coincidence, since we are defining the opposite construction from $\tilde{X}$: instead of starting with a graph of groups and obtaining a graph with an action, we start with an action

on a graph and are defining the corresponding graph of groups.
The choice we made here in saying that $\mathrm{o}(jy) \in jT$ is the same choice as we made in the definition of $\tilde{X}$ by saying that $\pi_{\tilde{y}} = G_w^w$. We will see that this is enough to make the two constructions inverse to each other (Theorem 4).

For $Q \in \mathrm{V}(X),\ z \in \mathrm{E}(X)$ let their stabilisers in $G$ be $G_Q,\ G_z$. We define $(\mathcal{G}, Y)$ by setting

$$\begin{aligned} G_P &= G_{jP} && \text{for } P \in \mathrm{V}(Y), \\ G_y &= G_{jy} && \text{for } y \in \mathrm{E}(Y), \\ G_y &\to G_{\mathrm{t}(y)} && \text{is } a \mapsto a^y = \gamma_y^{\sigma(y)-1} a\, \gamma_y^{1-\sigma(y)}. \end{aligned}$$

The last expression makes sense since

$$G_y = G_{jy} \leqslant G_{\mathrm{t}(jy)} = G_{\gamma_y^{1-\sigma(y)} j\mathrm{t}(y)} = \gamma_y^{1-\sigma(y)} G_{\mathrm{t}(y)} \gamma_y^{\sigma(y)-1},$$

hence

$$\gamma_y^{\sigma(y)-1} G_y \gamma_y^{1-\sigma(y)} \leqslant G_{\mathrm{t}(y)}.$$

Then if $\pi = \pi_1(\mathcal{G}, Y, T)$, we can define the homomorphism

$$\begin{aligned} \phi : \pi &\to G \\ G_P &\hookrightarrow G \\ g_y &\mapsto \gamma_y \end{aligned}$$

$\phi$ is well-defined since for $y \in T$, $\gamma_y = 1$, and for $a \in G_y$,

$$\gamma_y a^y \gamma_y^{-1} = \gamma_y^{\sigma(y)} a \gamma_y^{-\sigma(y)} = \gamma_{\bar{y}}^{\sigma(\bar{y})-1} a\, \gamma_{\bar{y}}^{1-\sigma(\bar{y})} = a^{\bar{y}}.$$

We can also define the map of graphs

$$\begin{aligned} \psi : \tilde{X}(\mathcal{G}, Y, T) &\to X \\ g\tilde{P} &\mapsto \phi(g) jP \\ g\tilde{y} &\mapsto \phi(g) jy \end{aligned}$$

To check that $\psi$ is a well-defined map of graphs, we need to check that for any edge $w = g\tilde{y}$ of $\tilde{X}$ we have $\psi(\bar{w}) = \overline{\psi(w)}$ and $\psi(\mathrm{o}(w)) = \mathrm{o}(\psi(w))$.
Indeed:

$$\psi(\bar{w}) = \psi(g\bar{\tilde{y}}) = \phi(g) j\bar{y} = \phi(g)\overline{jy} = \overline{\phi(g) jy} = \overline{\psi(w)}$$

and

$$\psi(\mathrm{o}(w)) = \psi(g g_y^{-\sigma(y)} \widetilde{\mathrm{o}(y)}) = \phi(g)\gamma_y^{-\sigma(y)} j\mathrm{o}(y) = \phi(g)\mathrm{o}(jy) = \mathrm{o}(\phi(g) jy) = \mathrm{o}(\psi(w)).$$

Further $\psi$ is $\phi$-equivariant in the sense that the following diagram commutes for all $g \in \pi$:

$$\begin{tikzcd} \tilde{X} \arrow[r, "g"] \arrow[d, "\psi"'] & \tilde{X} \arrow[d, "\psi"] \\ X \arrow[r, "\phi(g)"'] & X \end{tikzcd}$$

Note that by definition of $\phi$, it maps $\pi_{\tilde{P}}$ isomorphically onto $G_{jP}$. Since the stabilisers of other vertices are related to these by conjugation, we see that for all vertices $Q$ of $\tilde{X}$, $\phi$ restricts to isomorphisms between stabilisers $\pi_Q$ (of $Q$ in $\tilde{X}$) and $G_{\psi(Q)}$ (of $\psi(Q)$ in $X$).

Now we shall show that $\phi$ is surjective, by constructing a subgroup $H$ of $\phi(\pi)$ for which we can show that $H = G$.
Let $z$ be an edge of $X$ with origin in $jT$. Then if it projects to $y$ in $Y$, there is a $g \in G$ such that $gz = jy$. Since $\mathrm{o}(z), \mathrm{o}(jy) \in jT$, $\mathrm{o}(z) = \mathrm{o}(jy)$, hence $g \in G_{\mathrm{o}(z)}$. Since $\mathrm{t}(jy) = \gamma_y^{1-\sigma(y)} j\mathrm{t}(y)$, the edge $\gamma_y^{\sigma(y)-1} jy$ ends in $jT$. So if $h_z = \gamma_y^{\sigma(y)-1} g$, then $h_z z$ ends in $jT$.

Let $H$ be the subgroup of $G$ generated by $\{G_{jP} : P \in \mathrm{V}(Y)\}$ and $\{h_z : \mathrm{o}(z) \in jT\}$. Then $H$ lies in $\phi(\pi)$, as $G_{jP} = \phi(\pi_{\tilde{P}})$ and $\gamma_y = \phi(g_y)$.

**Lemma 4.** If $H$ is as defined above, then $H = G$. Hence $\phi$ is surjective.

*Proof.* As $\mathrm{V}(X) = \bigsqcup_{g \in G} \mathrm{V}(g \cdot jT)$, it suffices to show that $H \cdot \mathrm{V}(jT) = \mathrm{V}(X)$.

Consider $W$, the smallest subgraph of $X$ containing $jy$ for all $y \in \mathrm{E}(Y)$. Then as $Y = G \backslash X$, $G \cdot W = X$.

We know that $\mathrm{V}(W) \subset H \cdot \mathrm{V}(jT)$: if $Q$ is a vertex of $W$ that is not a vertex of $jT$, then there is an edge $z$ such that $\mathrm{o}(z) \in jT$, $\mathrm{t}(z) = Q$. But $h_z \in H$ and $h_z Q = \mathrm{t}(h_z z) \in jT$.
Hence it suffices to show that $H \cdot W = X$. As $X$ is connected, it is enough to show that every edge of $X$ starting in $H \cdot W$ is contained in $H \cdot W$.

Let $w$ be an edge of $X$ with $\mathrm{o}(w) \in H \cdot W$. We can translate by an element of $H$ to make sure that $Q = \mathrm{t}(w)$ lies in $jT$. Since $G \cdot W = X$, there is a $g \in G$ such that $z = gw$ lies in $W$. We shall show that $g \in H$, this implies that $w \in g^{-1}W \subset H \cdot W$.

Since $z \in W$, either $\mathrm{o}(z) \in jT$ or $\mathrm{t}(z) \in jT$.
Case (i): If $\mathrm{o}(z) \in jT$, $h_z \in H$ and $h_z z$ ends in $jT$, so $h_z \mathrm{t}(z) = h_z g Q$ and $Q$ both lie in $jT$. This implies that $Q = h_z g Q$, so $g \in h_z^{-1} G_Q \subset H$.
Case (ii): If $\mathrm{t}(z) \in jT$, then $\mathrm{t}(gw) = Q = \mathrm{t}(w)$, so $g \in G_Q \subset H$. □

**Definition 13.** A map of graphs $f$ is **locally injective** if for every vertex $P$ it is injective on the set of edges with origin at $P$.

**Lemma 5.** $\psi$ is locally injective.

*Proof.* For two edges $z_1 = g_1 \tilde{y}_1,\ z_2 = g_2 \tilde{y}_2$ of $\tilde{X}$, if $\psi(z_1) = \psi(z_2)$, then $jy_1$ and $jy_2$ belong to the same $G$-orbit in $X$, so $y_1 = y_2$.

Then consider two edges $z_1 = g_1 \tilde{y}, z_2 = g_2 \tilde{y}$ with the same origin $Q$. If $\psi(z_1) = \psi(z_2)$, then $\phi(g_1) = \phi(g_2)$. Now, as $\mathrm{o}(g_1 \tilde{y}) = \mathrm{o}(g_2 \tilde{y}) = Q$, $g_1^{-1} g_2 \in G_Q$. But $\phi$ restricts to an isomorphism on $G_Q$ and $\phi(g_1^{-1} g_2) = 1$, hence $g_1 = g_2$, so $z_1 = z_2$.

Thus $\psi$ is locally injective. □

**Theorem 4.** Suppose $G$ acts on a connected non-empty graph $X$. Let $Y = G\backslash X$, $T$ a maximal tree of $Y$. Define $(\mathcal{G}, Y, T)$, $\tilde{X}$ as above, along with the maps $\psi$ and $\phi$. Then the following are equivalent:

(1) $X$ is a tree.

(2) $\psi : \tilde{X} \to X$ is an isomorphism

(3) $\phi : \pi_1(\mathcal{G}, Y, T) \to G$ is an isomorphism.

*Proof.*

(2) $\implies$ (1): We know from Theorem 3 that $\tilde{X}$ is a tree.

(3) $\implies$ (2): If $\psi(g_1\tilde{P}_1) = \psi(g_2\tilde{P}_2)$, then $\phi(g_1)jP_1 = \phi_2 jP_2$, so $jP_1$ and $jP_2$ are in the same $G$-orbit, hence $P_1 = P_2$. Then as $\phi$ is an isomorphism, $g_1 = g_2$. Hence $\psi$ is injective, and so as it is also surjective (Lemma 4), it is an isomorphism.

(1) $\implies$ (2): We know from Lemmas 4 and 5 that $\psi$ is surjective and locally injective. Hence it suffices to show that a locally injective map into a tree is injective.
Indeed, suppose $Q_1 \neq Q_2$ but $\psi(Q_1) = \psi(Q_2)$. As $\tilde{X}$ is connected, there is a path $c = (z_1, ..., z_n)$ without backtracking from $Q_1$ to $Q_2$ in $\tilde{X}$. The $\psi(c)$ is a closed path in $X$, so as $X$ is a tree, it must have a backtracking $\psi(z_{i+1}) = \overline{\psi(z_i)}$. But then $\psi$ is not locally injective at $Q' = \mathrm{t}(z_i)$ as $\bar{z}_i$ and $z_{i+1}$ are distinct edges with origin at $Q'$ that have the same image under $\psi$. We get a contradiction, hence $\psi$ is injective.

(2) $\implies$ (3): Let $N = \ker\phi$, $P$ a vertex of $Y$. We know that $\phi$ restricts to an isomorphism on $G_{\tilde{P}}$, so $G_{\tilde{P}} \cap N = \{1\}$. Suppose $n \in N$, $n \neq 1$. Then $\tilde{P} \neq n\tilde{P}$ in $\tilde{X}$ but $\psi(\tilde{P}) = jP = \psi(n\tilde{P})$. Hence if $\phi$ is not an isomorphism, then $\psi$ is not either. $\square$

Note that the implication (1) $\implies$ (3) gives a structure theorem for groups acting on trees: $G$ is generated by the stabilisers $G_P$ for $P \in \mathrm{V}(Y)$ and the elements $\gamma_y$ for $y \in \mathrm{E}(Y)$, subject to relations $\gamma_y = 1$ for $y \in T$ and $\gamma_y a^y \gamma_y^{-1} = a^{\bar{y}}$ for an appropriate definition of the inclusions $a \mapsto a^y$.

*Example:* Let $X$ be the tree with vertices $\mathbb{Z} \times \{-1, 0, +1\}$, and positively oriented edges from $(n, 0)$ to $(n+1, 0)$, $(n, -1)$, $(n, +1)$ for all $n \in \mathbb{Z}$.

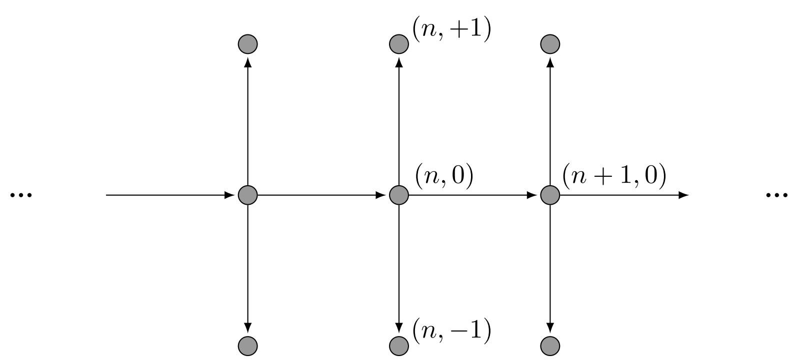



Figure 9: Tree $X$.

Let $s$ be the shift automorphism of $X$, mapping the vertex $(n, b)$ onto $(n+1, b)$.
Let $r$ be the reflection automorphism, which switches $(0, +1)$ and $(0, -1)$, and fixes all other vertices.

Then $G = \langle s, r \rangle$ acts on $X$ without inversion. Then $Y = G\backslash X$ has two vertices $P$ and $Q$ (corresponding to the orbits of $jP = (0, 0)$ and $jQ = (0, +1)$), and two edges, $y$ and $z$ (corresponding to the orbits of $jy$ from $jP$ to $(1, 0)$ and $jz$ from $jP$ to $jQ$). The maximal tree $T$ of $Y$ consists of $P, Q$ and $z$.

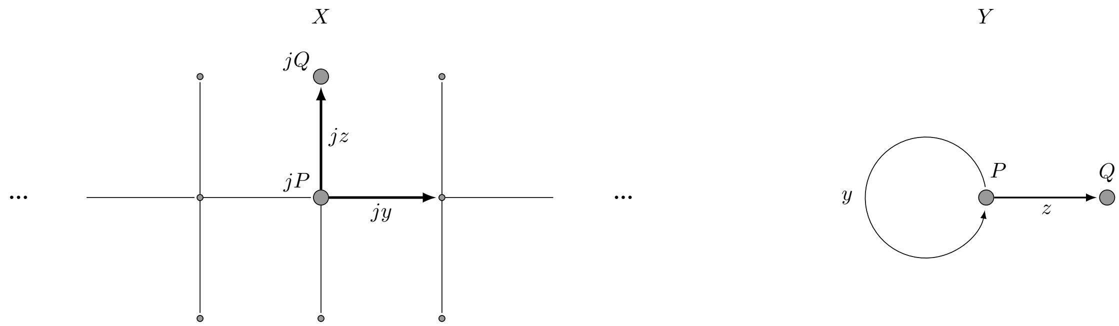



Figure 10: The quotient graph $Y$ and the section $j$.

If $r_n = s^n r s^{-n}$ is the automorphism that interchanges $(n, +1)$ and $(n, -1)$, then we can see that the stabiliser of $jP$ (and also of $jy$) is the Cartesian product $\times_{n\in\mathbb{Z}} \langle r_n \rangle$. Similarly, the stabiliser of $jQ$ and also of $jz$ is $\times_{n\in\mathbb{Z},\ n\neq 0} \langle r_n \rangle$.

We can see that $\gamma_y = s$, and the maps $a \mapsto a^y,\ a \mapsto a^{\bar{y}}$ are defined by $r_n^y = r_{n-1},\ r_n^{\bar{y}} = r_n$.

Hence by Theorem 4 we see that another presentation for $G$ is

$$G = \langle \gamma_y,\ r_n \text{ for } n \in \mathbb{Z} \,|\, r_n^2 = 1, \gamma_y r_n \gamma_y^{-1} = r_{n+1} \rangle.$$

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